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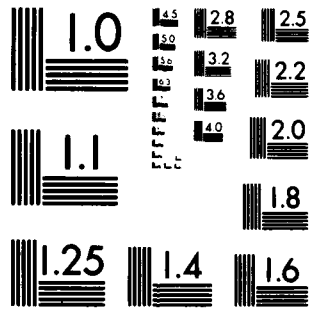
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SENSITIVITY ANALYSIS OF A NONLINEAR
STRUCTURAL DESIGN PROBLEM.

by

Anthony V. Fiacco
Abolfazl/Ghaemi

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⑥ SENSITIVITY ANALYSIS OF A NONLINEAR
STRUCTURAL DESIGN PROBLEM.

by

⑩ Anthony V. Fiacco
Abolfazl/Ghaemi

⑨ Sensitivity

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Abolfazl Ghaemi

This paper presents a user-oriented demonstration of practical applicability of a recently developed sensitivity analysis methodology. The required underlying theory for sensitivity analysis of nonlinear programming problems and an algorithm for implementing the subsequent results of this theory using the Sequential Unconstrained Minimization Technique are briefly reviewed. A recently developed computer program based on this approach is used to conduct a sensitivity analysis of a nonlinear programming problem associated with the optimization of the design of a vertically corrugated transverse bulkhead of an oil tanker. Major assumptions and results associated with the basic sensitivity analysis approach are verified for the subject problem. First order sensitivity information is used to estimate measures associated with various perturbations of the given problem data. Details of stability and convergence characteristics of the solution points and their partial derivatives with respect to the problem parameters are amply tabulated and graphically illustrated.

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1. INTRODUCTION

The purpose of the current paper is to conduct a first order sensitivity analysis of a nonlinear structural design problem discussed in Chapter 6 of the book by Bracken and McCormick [8].

By sensitivity analysis is meant an analysis of the effect on the optimal objective function value and on an optimal solution point of small perturbations in the model parameters. The importance of such an analysis in real world optimization problems cannot be overstated. It provides the model maker and user with invaluable information regarding the functional relationship between a solution and the design parameters. This has many potential applications. For example, identification of those parameters having the most significant impact on the optimal solution can provide a basis for developing educated guidelines for taking appropriate and efficient action toward effecting parameter changes that will give an optimal marginal improvement of system design or performance.

A theoretical basis for sensitivity analysis for nonlinear programming was given by Fiacco and McCormick [13] and generalized and extended by Fiacco [10]. Based on the approach given in [10], an algorithmic procedure

was proposed by Fiacco and implemented by Causey [9]. Later on it was refined and recoded by Mylander [14], using the several subroutines of the SUMT-version 4 computer code by Mylander, Holmes, and McCormick [15], which implements the Sequential Unconstrained Minimization Technique for nonlinear programming using the logarithmic-quadratic-loss penalty function. This will be described in Section 2. Mylander's sensitivity subroutines were integrated with the SUMT-version 4 computer program by Armacost and Mylander [7]. This routine was further revised and expanded by Armacost [1].

The latter version of the routine, now called "SENSUMT," is compiled in The George Washington University Computer Center and constitutes the main element on which the present sensitivity analysis is based. The study is conducted in three phases: Phase 1 deals with the solution of the problem. Phase 2 deals with the solution and sensitivity analysis study of the problem when each of the right hand sides of the constraints is perturbed. Several problems are analyzed, each associated with a significantly different right hand side initial value. Phase 3 deals with a sensitivity analysis study with respect to each problem design parameter, again for several problems, each associated with a significantly different set of problem design parameter initial values.

Estimates are given of the optimal solution of this family of problems, using the optimal solution and first order sensitivity information of the unperturbed problem. Furthermore, several stability and convergence characteristics of the solution points and their partial derivatives with respect to the problem parameters are computationally verified.

As an example of the numerous inferences that can be made, the analysis reveals that the weight of the optimally designed structure is most sensitive to the minimum allowable structural thickness and, next, to a corrosion allowance, two of the many design parameters of the problem.

Similar studies have been conducted by Armacost and Fiacco on a variety of problems, including a cattle feed problem [2] and a multi-item inventory problem [5]; and by the authors on stream-water pollution-abatement model involving numerous parameters [11], [12].

2. BASIC SENSITIVITY RESULTS

A comprehensive review of the sensitivity results obtained for nonlinear programming problems is outside the scope of the present paper. In the following we review only the theory supporting the validity of the computational results obtained. The basic result was given in Fiacco and McCormick [13, Theorem 6] for a particular class of perturbations. Fiacco generalized the theory and also established a theoretical basis for utilizing a penalty function method to estimate sensitivity information associated with a local solution and its associated optimal Lagrange multipliers, for a large class of nonlinear programming problems, with respect to a general parametric variation in the problem functions [10].

Armacost and Fiacco subsequently computationally implemented this approach to demonstrate practical applicability [2] and applied this theory to obtain the first and second order sensitivity results for the optimal value function, deriving formulas for the efficient calculation of the sensitivities of this function, as well as the sensitivities of the local solution point and its associated optimal Lagrange multipliers [3], [4].

The parametric mathematical programming problem considered by Fiacco is of the following general form:

$$\begin{aligned} &\text{minimize } f(x, \epsilon) \\ &x \in E^n \\ &\text{subject to } g_i(x, \epsilon) \geq 0, \quad i = 1, \dots, m, \quad P(\epsilon) \\ &\quad h_j(x, \epsilon) = 0, \quad j = 1, \dots, p, \end{aligned}$$

where x is the usual vector of variables and ϵ is a k -component vector of numbers called "parameters." It is desired to analyze the behavior of a solution vector $x(\epsilon)$ and the optimal solution value $f^*(\epsilon) \equiv f[x(\epsilon), \epsilon]$ near some given value of ϵ . Without loss of generality, assume that the parameter vector of interest is $\epsilon = 0$.

The Lagrangian for Problem $P(\epsilon)$ is defined as

$$L(x, u, w, \epsilon) \equiv f(x, \epsilon) - \sum_{i=1}^m u_i g_i(x, \epsilon) + \sum_{j=1}^p w_j h_j(x, \epsilon).$$

The sensitivity results are based on the following four assumptions:

- A1 - The functions defining Problem $P(\epsilon)$ are twice continuously differentiable in (x, ϵ) in a neighborhood of $(x^*, 0)$.
- A2 - The second order sufficient conditions for a local minimum of Problem $P(0)$ hold at x^* with associated Lagrange multipliers u^* and w^* .
- A3 - The gradients $\nabla_x g_i(x^*, 0)$, for all i such that $g_i(x^*, 0) = 0$, and $\nabla_x h_j(x^*, 0)$, $j = 1, \dots, p$ are linearly independent.
- A4 - Strict complementary slackness holds at x^* when $\epsilon = 0$ (i.e., $u_i^* > 0$ for all i such that $g_i(x^*, 0) = 0$).

Under the above assumptions, Fiacco [10] established the following generalization of Theorem 6 in [13].

Lemma 2.1 (Local characterization of a Kuhn-Tucker triple.) If assumption A1, A2, A3 and A4 hold for Problem $P(\epsilon)$ at $(x^*, 0)$, then

- (a) x^* is a local isolated minimizing point of Problem $P(0)$ and the associated Lagrange multipliers u^* and w^* are unique;
- (b) for ϵ in a neighborhood of 0, there exists a unique once continuously differentiable vector function $y(\epsilon) = (x(\epsilon), u(\epsilon), w(\epsilon))^T$ satisfying the second order sufficient conditions for a local minimum of Problem $P(\epsilon)$ such that $y(0) = (x^*, u^*, w^*)^T = y^*$ and hence, $x(\epsilon)$ is a locally unique, local minimum of Problem $P(\epsilon)$ with associated unique Lagrange multipliers $u(\epsilon)$ and $w(\epsilon)$; and
- (c) for ϵ near 0, the set of binding inequalities is unchanged, strict complementary slackness continues to hold, and the binding constraint gradients are linearly independent at $x(\epsilon)$.

- (d) (Fiacco and Armacost [3]), for ϵ near 0, the gradient of the optimal value function is

$$\nabla_{\epsilon} f^*(\epsilon) = \nabla_{\epsilon} L(u, \epsilon) \quad \text{at } y = y(\epsilon),$$

- (e) which also means that, for ϵ near 0, the Hessian of the optimal value function is

$$\nabla_{\epsilon}^2 f^*(\epsilon) = \nabla_{\epsilon}^2 L(y(\epsilon), \epsilon).$$

The above results provide a characterization of a local solution of Problem $P(\epsilon)$ and its associated optimal Lagrange multipliers near $\epsilon = 0$. They show that the Kuhn-Tucker triple $y(\epsilon)$ is unique and well behaved, under the given conditions. Since $y(\epsilon)$ is once differentiable, the partial derivatives of the components of $y(\epsilon)$ are well defined. This fact and Assumption A1 also mean that the functions defining Problem $P(\epsilon)$ are once continuously differentiable functions of ϵ along the "solution trajectory" $x(\epsilon)$ near $\epsilon = 0$, and the Lagrangian is a once continuously differentiable function of ϵ along the "Kuhn-Tucker point trajectory." The above results constitute the structure for numerous developments and extensions, many of which have been established by Fiacco [10] and Armacost and Fiacco [2-6].

The realization of this theorem for the parametric right hand side problem of special interest in the present study is treated in detail by Armacost and Fiacco [1]. The parametric right hand side problem is the following important realization of $P(\epsilon)$:

$$\begin{aligned} &\text{minimize} && f(x) \\ &\text{subject to} && g_i(x) \geq \epsilon_i && i = 1, \dots, m && R(\epsilon) \\ &&& h_j(x) = \epsilon_{j+m} && j = 1, \dots, p. \end{aligned}$$

The Lagrangian for $R(\epsilon)$ is

$$L(x, u, w) \equiv f(x) - \sum_{i=1}^m u_i (g_i(x) - \epsilon_i) + \sum_{j=1}^p w_j (h_j(x) - \epsilon_{j+m}).$$

As evident from the Lagrangian, the results (d) and (e) of Lemma 2.1 for problem $R(\epsilon)$ simplify respectively to

$$(d') \quad \nabla_{\epsilon} f^*(\epsilon) = \begin{bmatrix} u(\epsilon) \\ -w(\epsilon) \end{bmatrix}^T, \text{ and}$$

$$(e') \quad \nabla_{\epsilon}^2 f^*(\epsilon) = \begin{bmatrix} \nabla_{\epsilon} u(\epsilon) \\ -\nabla_{\epsilon} w(\epsilon) \end{bmatrix}.$$

Fiacco [10] has shown that the class of algorithms based on twice continuously differentiable penalty functions (specifically, using the logarithmic-quadratic loss penalty function) can be used to estimate $y(\cdot)$ and its derivatives in a neighborhood of $\epsilon = 0$, for the general problem $P(\epsilon)$. Minimization of the penalty function with penalty parameter r yields a solution of a perturbation of the Kuhn-Tucker system in a neighborhood of $(\epsilon, r) = (0, 0)$. Armacost and Fiacco [3] define an optimal value penalty function and obtain first- and second-order sensitivity estimates which converge to the corresponding sensitivities for the optimal value function for Problem $P(\epsilon)$.

The logarithmic-quadratic penalty function is

$$W(x, \epsilon, r) = f(x, \epsilon) - r \sum_{i=1}^m \ln g_i(x, \epsilon) + (1/2r) \sum_{j=1}^p h_j^2(x, \epsilon).$$

Lemma 2.2 (Fiacco [10, Theorem 3.1]). If the assumptions $A_1 - A_4$ hold, then in a neighborhood of $(\epsilon, r) = (0, 0)$ there exists a unique once continuously differentiable vector function $y(\epsilon, r) = [x(\epsilon, r), u(\epsilon, r), w(\epsilon, r)]^T$ satisfying

$$\begin{aligned} \nabla_x L(x, u, w, \epsilon) &= 0, \\ u_i g_i(x, \epsilon) &= r, \quad i=1, \dots, m, \\ h_j(x, \epsilon) &= w_j r, \quad j=1, \dots, p, \end{aligned} \quad (2.1)$$

with $y(0, 0) = (x^*, u^*, w^*)$, and such that for any (ϵ, r) near $(0, 0)$ and $r > 0$, $x(\epsilon, r)$ is a locally unique unconstrained local minimizing point of $W(x, \epsilon, r)$, with $g_i[x(\epsilon, r), \epsilon] > 0$, $i=1, \dots, m$, and $\nabla_x^2 W[x(\epsilon, r), \epsilon, r]$ positive definite.

The relevance of equations (2.1) is the fact that, under the given conditions, when $r = 0$, they are necessary conditions that must hold at a

local solution of $P(0)$ and, with $r > 0$, they are necessary conditions for an unconstrained minimum of $W(x, \epsilon, r)$. The latter fact can be made obvious by solving for u_i and w_j in (2.1) and obtaining

$$\begin{aligned}\nabla_x L(x, u, w, \epsilon) &\equiv \nabla_x f - \sum u_i \nabla_x g_i + \sum w_j \nabla_x h_j \\ &= \nabla_x f - r \sum (1/g_i) \nabla_x g_i + (1/r) \sum h_j \nabla_x h_j \\ &= \nabla_x W(x, \epsilon, r) .\end{aligned}$$

Thus, if $y(\epsilon, r)$ is a solution of (2.1), then

$$\nabla_x W[x(\epsilon, r), \epsilon, r] = \nabla_x L[x(\epsilon, r), u(\epsilon, r), w(\epsilon, r), \epsilon] = 0 . \quad (2.2)$$

This explicit connection between the optimality conditions of local solutions of $P(\epsilon)$ and unconstrained minima of $W(x, \epsilon, r)$ makes it possible to approximate information characterizing a local solution of $P(\epsilon)$ by algorithmic calculations associated with utilizing $W(x, \epsilon, r)$ to solve $P(\epsilon)$. In particular, differentiating (2.2) with respect to ϵ yields

$$\nabla_x^2 W \nabla_\epsilon x + \nabla_{\epsilon x}^2 W = 0$$

and using the fact that $\nabla_x^2 W$ is positive definite (a conclusion of Lemma 2.2) yields

$$\nabla_\epsilon x = -\nabla_x^2 W^{-1} \nabla_{\epsilon x}^2 W \quad (2.3)$$

evaluated of course at $x(\epsilon, r)$. Given $\nabla_\epsilon x(\epsilon, r)$, the derivatives of the multipliers, $\nabla_\epsilon u_i(\epsilon, r)$ and $\nabla_\epsilon w_j(\epsilon, r)$, can then be calculated by differentiating the last two systems of equations of (2.1) at $x(\epsilon, r)$ with respect to ϵ .

Lemma 2.3 (Fiacco [10]). For ϵ in a neighborhood of $\epsilon = 0$, it follows that:

- (a) $\lim_{r \rightarrow 0^+} y(\epsilon, r) = y(\epsilon, 0) = y(\epsilon)$, the Kuhn-Tucker triple characterized in conclusion (b) of Lemma 2.1, and
- (b) $\lim_{r \rightarrow 0^+} \nabla_\epsilon y(\epsilon, r) = \nabla_\epsilon y(\epsilon, 0) = \nabla_\epsilon y(\epsilon)$.

Under the conditions of Lemma 2.1, $x(\epsilon, r)$ is a locally unique minimizing point of $W(x, \epsilon, r)$. Define the "optimal value penalty function" as

$$W^*(\epsilon, r) = W(x(\epsilon, r), \epsilon, r).$$

Armstrong and Fiacco [2, Theorem 4 and Corollary 4.1] have obtained the further results useful for estimating the first- and second-order sensitivity of the optimal value function $f^*(\epsilon)$ of Problem $P(\epsilon)$.

Lemma 2.4 (Armstrong and Fiacco [3]) If the assumptions $A_1 - A_4$ hold for Problem $P(\cdot)$, then in a neighborhood of $\epsilon = 0$,

- (a) $\lim_{r \rightarrow 0^+} W^*(\epsilon, r) = f^*(\epsilon)$,
- (b) $\nabla_{\epsilon} W^*(\epsilon, r) = \nabla_{\epsilon} L(y, \epsilon)$ at $y = y(\epsilon, r)$,
- (c) $\lim_{r \rightarrow 0^+} \nabla_{\epsilon} W^*(\epsilon, r) = \nabla_{\epsilon} f^*(\epsilon)$,
- (d) $\nabla_{\epsilon}^2 W^*(\epsilon, r) = \nabla_{\epsilon}^2 L(y(\epsilon, r), \epsilon)$

and

- (e) $\lim_{r \rightarrow 0^+} \nabla_{\epsilon}^2 W^*(\epsilon, r) = \nabla_{\epsilon}^2 f^*(\epsilon)$,

where convergence is component by component in all cases.

Lemmas 2.1 - 2.4 enable us to calculate an estimate of $y(\epsilon)$, $\nabla_{\epsilon} y(\epsilon)$, $\nabla_{\epsilon} f^*(\epsilon)$, and $\nabla_{\epsilon}^2 f^*(\epsilon)$ when ϵ is near 0 and r is near 0, once $y(\epsilon, r)$ is available.

In the next section we briefly present the algorithmic implementation of some of the above results.

3. THE ALGORITHM

The penalty function algorithm SUMT estimates the solution of the general mathematical problem $P(\epsilon)$ by estimating the unconstrained minima of the penalty function $W(x, \epsilon, r)$ at successively decreasing values of the penalty function parameter $r > 0$. Under conditions weaker than those assumed here, Fiacco and McCormick [13] have shown that as r approaches zero, the sequence of the unconstrained minima of $W(x, \epsilon, r)$ will approach a solution of $P(\epsilon)$. Each

unconstrained penalty function minimization is thus a "subproblem" associated with a particular value of the penalty function parameter r .

The successive steps of the algorithm for sensitivity analysis that are presented in reference [1] are listed below. The notation \bar{x} or $\bar{x}(\bar{\epsilon})$ denotes the estimate of a solution point of $P(\bar{\epsilon})$ calculated by SUMT for a given value of the penalty function parameter r , where $\bar{\epsilon}$ denotes the value of the problem parameter for which this sensitivity is estimated.

Step 1: Compute a representation of $\nabla_x^2 W^{-1} = \nabla_x^2 W(\bar{x}, \bar{\epsilon}, r)^{-1}$ by L-U decomposition using the SUMT subroutines. If $\nabla_x^2 W$ is not positive definite, terminate the sensitivity analysis.

Step 2: Estimate $\partial(\nabla_x W^T)/\partial \epsilon_j$ using the central differencing formula

$$\partial(\nabla_x W^T)/\partial \epsilon_j \doteq (1/2\Delta) (\nabla_x W(\bar{x}, \bar{\epsilon} + \Delta \epsilon_j, r)^T - \nabla_x W(\bar{x}, \bar{\epsilon} - \Delta \epsilon_j, r)^T) .$$

(Alternately, the matrix of partial derivatives could be determined analytically from $W(x, \epsilon, r)$.)

Step 3: Calculate

$$\partial \bar{x}(\bar{\epsilon})/\partial \epsilon_j = -\nabla_x^2 W^{-1} \partial(\nabla_x W^T)/\partial \epsilon_j .$$

Step 4: Estimate $\nabla_{\epsilon} g_1(\bar{x}, \bar{\epsilon})$ and $\nabla_{\epsilon} h_1(\bar{x}, \bar{\epsilon})$ using

$$\partial g_1(\bar{x}, \bar{\epsilon})/\partial \epsilon_j \doteq (1/2\Delta) (g_1(\bar{x}, \bar{\epsilon} + \Delta \epsilon_j) - g_1(\bar{x}, \bar{\epsilon} - \Delta \epsilon_j)) , \text{ and}$$

$$\partial h_1(\bar{x}, \bar{\epsilon})/\partial \epsilon_j \doteq (1/2\Delta) (h_1(\bar{x}, \bar{\epsilon} + \Delta \epsilon_j) - h_1(\bar{x}, \bar{\epsilon} - \Delta \epsilon_j)) .$$

Step 5: Estimate the components of $\nabla_{\epsilon} u(\bar{\epsilon})$ for $i = 1, \dots, m$ using

$$\begin{aligned} \partial u_1(\bar{\epsilon})/\partial \epsilon_j &= -(r/g_1(\bar{x}, \bar{\epsilon})^2) (\nabla_x g_1(\bar{x}, \bar{\epsilon}) \partial \bar{x}(\bar{\epsilon})/\partial \epsilon_j \\ &\quad + \partial g_1(\bar{x}, \bar{\epsilon})/\partial \epsilon_j) . \end{aligned}$$

Step 6: Estimate the components of $\nabla_{\epsilon} w(\bar{\epsilon})$ for $i = 1, \dots, p$ using

$$\begin{aligned} \partial w_1(\bar{\epsilon})/\partial \epsilon_j &= (1/r) (\nabla_x h_1(\bar{x}, \bar{\epsilon}) \partial \bar{x}(\bar{\epsilon})/\partial \epsilon_j \\ &\quad + \partial h_1(\bar{x}, \bar{\epsilon})/\partial \epsilon_j) . \end{aligned}$$

There are two methods for estimating $V_{\epsilon} f^*(\bar{\epsilon})$; the first using $V_{\epsilon} f^* = V_{\epsilon} f V_{\epsilon} x + V_{\epsilon} f$, with $V_{\epsilon} x$ obtained from Step 3, and the second method using the gradient of the penalty function W , or equivalently, the Lagrangian taken with respect to the parameters (Lemma 2.1, conclusion (d), Lemma 2.4, conclusion (b)). Both are incorporated in the computer program but used for different purposes. The former method gives the most accurate estimate of $V_{\epsilon} f^*(\bar{\epsilon})$ and is summarized in Steps 7 and 8.

Step 7: Estimate the components of $V_{\epsilon} f(\bar{x}, \bar{\epsilon})$ using the central differencing formula

$$\partial f(\bar{x}, \bar{\epsilon}) / \partial \epsilon_j \doteq (1/2\Delta)(f(\bar{x}, \bar{\epsilon} + \Delta e_j) - f(\bar{x}, \bar{\epsilon} - \Delta e_j)) .$$

Step 8: Calculate an estimate of the components of $V_{\epsilon} f^*(\bar{\epsilon})$ using the results of Steps 3 and 7 as

$$\partial f^*(\bar{\epsilon}) / \partial \epsilon_j \doteq V_x f(\bar{x}, \bar{\epsilon}) \partial \bar{x}(\bar{\epsilon}) / \partial \epsilon_j + \partial f(\bar{x}, \bar{\epsilon}) / \partial \epsilon_j .$$

The second method, using the gradient of the Lagrangian to estimate $V_{\epsilon} f^*(\bar{\epsilon})$, is computationally less expensive and is used to obtain rough estimates that single out the more crucial parameters for further analysis. This approximation is calculated as follows.

Step 9. Estimate the components of $V_{\epsilon} f^*(\bar{\epsilon})$ using the results of Steps 4 and 7 as

$$\begin{aligned} \partial f^*(\bar{\epsilon}) / \partial \epsilon_j &\doteq \partial f(\bar{x}, \bar{\epsilon}) / \partial \epsilon_j \\ &\quad - \sum_{i=1}^m u_i(\bar{\epsilon}, r) \partial g_i(\bar{x}, \bar{\epsilon}) / \partial \epsilon_j \\ &\quad + \sum_{i=1}^p w_i(\bar{\epsilon}, r) \partial h_i(\bar{x}, \bar{\epsilon}) / \partial \epsilon_j . \end{aligned}$$

4. PROBLEM DESCRIPTION

The nonlinear programming model for the optimization of the design of a vertically corrugated transverse bulkhead of an oil tanker, discussed in Chapter 6 of [8], was selected for the sensitivity analysis. The material in this section follows the presentation in [8] rather closely.

Vertical transverse bulkheads form the lateral walls of the internal compartments of the tankers that hold liquid cargo. Figure 1 shows three different views of a corrugated bulkhead, consisting of a bottom, middle, and top panel, fastened together along the stringers EF and CD. It is assumed that the shape of the corrugations are identical in all panels but their thicknesses are allowed to vary from panel to panel. The lengths of the top, middle, and bottom panels denoted by l_t , l_m , and l_b , respectively, are fixed, as is the common width B of the panels.

The problem design variables are indicated in Figure 2 and defined as follows:

- b_1 = width of the flange
- b_2 = length of the web
- d = depth of corrugation
- t_t = thickness of top panel
- t_m = thickness of middle panel
- t_b = thickness of bottom panel.

The width of a corrugation s is depicted and used in the model for convenience. It is not a design variable, being determined once the indicated variables are specified. In fact, it is easy to see that $s = b_1 + (b_2^2 - d^2)^{1/2}$.

The design parameters involved in the formulation of the model and their specified values are listed in Table 1. Those characterizing the shape of the bulkhead are depicted in Figures 1 and 3. Figure 3 also indicates the load configuration which will enter into the constraints of the model to be developed subsequently.

Objective Function

It is desired to determine values for these design variables that minimize the weight of the corrugated bulkhead subject to satisfying a number of constraints to be considered subsequently. The total weight w of the panels is easily calculated to be

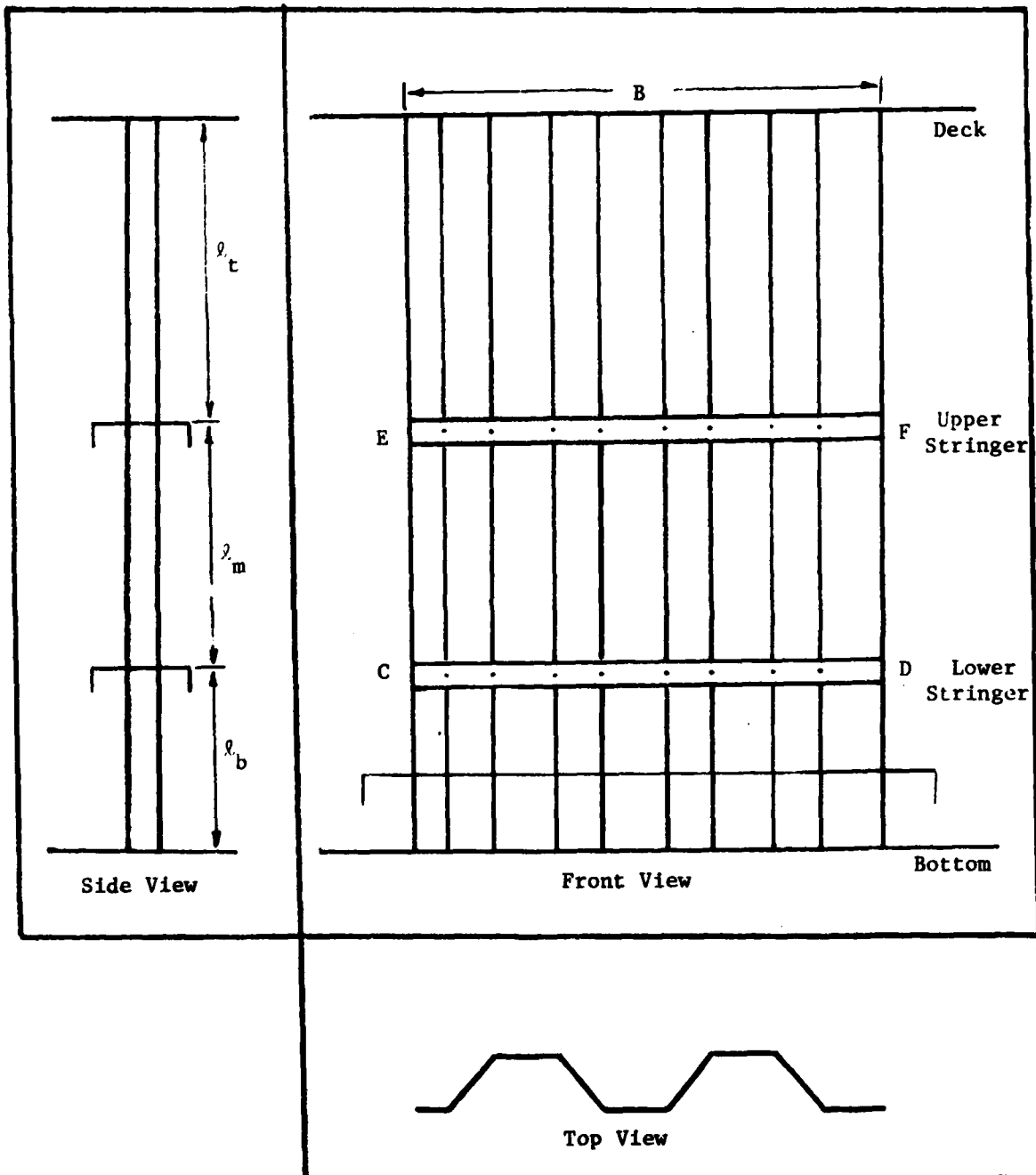


Figure 1. Vertical corrugated transverse bulkhead.

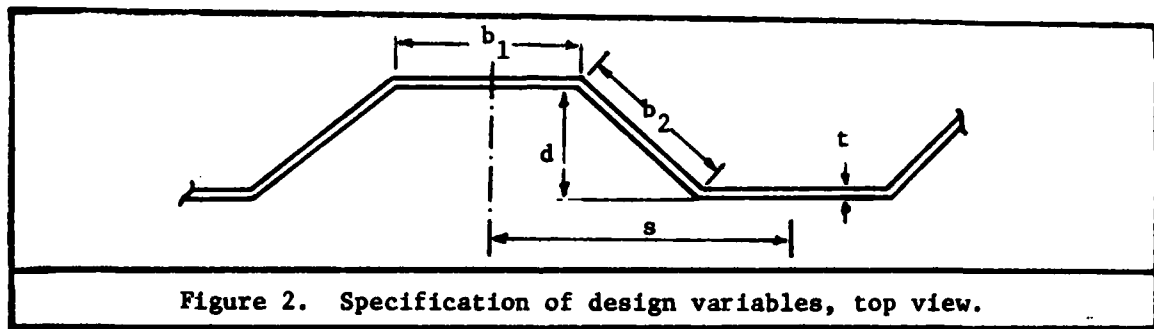


Table 1
PROBLEM DESIGN PARAMETERS
(MODEL INPUT DATA)

No.	Parameter	Meaning	Value
1	Γ	Weight per unit volume of the material	7.85 ton/cm^3
2	B	Width of the panel	476 cm
3	l_t	Length of the top panel	495 cm
4	l_m	Length of the middle panel	385 cm
5	l_b	Length of the bottom panel	315 cm
6	h_a	Distance between free liquid level and top of the structure	250 cm
7	h_t	Distance between free liquid level and middle of the top panel	498 cm
8	h_m	Distance between free liquid level and middle of the middle panel	938 cm
9	h_b	Distance between free liquid level and middle of the bottom panel	1288 cm
10	h_{1t}	Distance between free liquid level and base of the top panel	745 cm
11	h_{1m}	Distance between free liquid level and base of the middle panel	1130 cm
12	h_{1b}	Distance between free liquid level and base of the bottom panel	1445 cm
13	t_t^{\min}	Minimum allowable thickness of the top panel	1.05 cm
14	t_m^{\min}	Minimum allowable thickness of the middle panel	1.05 cm
15	t_b^{\min}	Minimum allowable thickness of the bottom panel	1.05 cm
16	e	Effectiveness of the flange (dimensionless)	0.8
17	k_1	Coefficient (function of maximum allowable bending stress)	$6.94 \times 10^{-8} \text{ cm}^{-1}$
18	k_2	Corrosion coefficient	0.15 cm

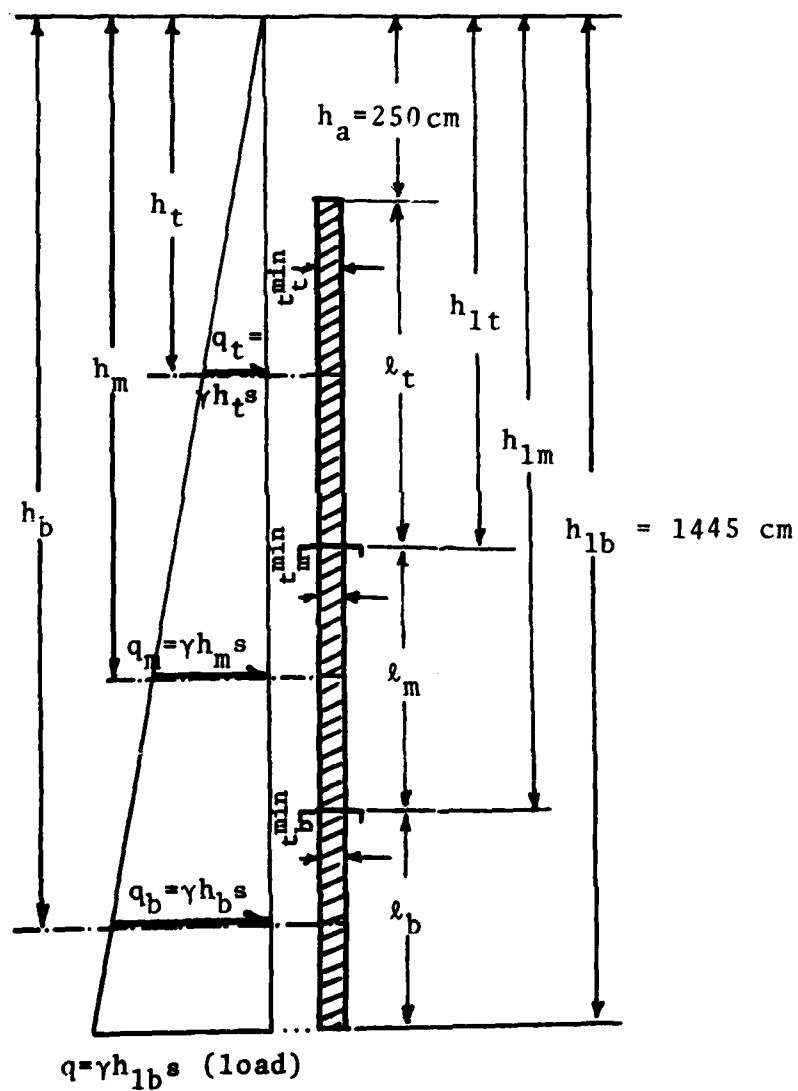


Figure 3. Specification of some of the design parameters and indication of load levels, side view.

$$w = \Gamma B(b_1 + b_2) \frac{t_t \ell + t_m \ell + t_b \ell}{s} \quad (1)$$

where Γ is the weight per unit volume of the material.

It is assumed that the design variables are to be given in centimeters.
 Γ will be given in tons per cubic centimeter.

Geometrical Constraint

As indicated in Figure 2, geometrical limitations suggest that the length of the web should at least be equal to the depth of the corrugation. Therefore, it is required that

$$b_2 - d \geq 0 \quad (2)$$

Bending Stress Constraint

Consider any panel of the type described and denote by h , the vertical distance from load level zero to the middle of the given panel. The bending stress at the supports of any such panel of thickness t and length ℓ is given by

$$\sigma = \frac{\frac{\gamma h s \ell^2}{12}}{\frac{td}{2} \left(\frac{b_2}{3} + b_1 e \right)} \quad (\text{kg/cm}^2)$$

where e (the effectiveness of the flange) is a dimensionless constant and $\gamma = .001 \text{ kg/cm}^3$ is the specific gravity of fresh water. (The numerator in the expression for σ is the bending moment at the supports in kg cm , and the denominator is the section modulus in cm^3 .)

The maximum permitted bending stress is given to be 1200 kg/cm^2 , thus requiring that

$$\sigma \leq 1200 (\text{kg/cm}^2) .$$

Therefore, the prior expression for σ and the above inequality result in the constraint

$$\frac{td}{2} \left(\frac{b_2}{3} + b_1 e \right) \geq k_1 h s \ell^2, \quad (3)$$

where $k_1 = \left(\frac{1}{12} \right) \left(\frac{Y}{1200} \right) = .694 \times 10^{-7} \text{ (cm}^{-1}\text{)}.$

Moment of Inertia Constraint

An additional constraint on the given panel is imposed as follows:

$$\left(\frac{d}{2} \right) \left(\frac{td}{2} \right) \left(\frac{b_2}{3} + b_1 e \right) \geq 2.2 (k_1 h s \ell^2)^{4/3} \quad (4)$$

where the expression on the left is the moment of inertia of the panel.

Thickness Requirement Constraint

For a given panel it is required that

$$t \geq \begin{cases} t_{\min} \\ (.39)(1.05 b_1) \sqrt{.01 h_1} + k_2 \\ (.39)(1.05 b_2) \sqrt{.01 h_1} + k_2 \end{cases} \quad (5)$$

where

t = plate thickness (centimeters)

t_{\min} = minimum allowable plate thickness (centimeters),
a function of ship length

h_1 = height to load level zero from the bottom of the
panel centimeters, and

k_2 = corrosion allowance (centimeters).

Natural Constraints

Since the design variables b_1 , b_2 , d , t , t_m and t_b are measures of length (dimensions of the corrugated bulkhead), they are constrained to be nonnegative.

Design parameters h , ℓ , t^{\min} and h_1 in constraints (3), (4), and (5) are subscripted by t , m and b to define the relative constraints for the top, middle, and bottom panels, respectively.

The Design Problem

Introducing variables x_1 through x_6 for the design variables b_1 , b_2 , d , t_t , t_m , t_b , respectively, and denoting the objective function w by f , yields the following nonlinear programming problem.

$$\text{minimize } f = l^B(x_1+x_2)(\ell_t x_4 + \ell_m x_5 + \ell_b x_6)[x_1 + (x_2^2 - x_3^2)^{\frac{1}{2}}]^{-1}$$

subject to

Constraint
Number

$$g_1 \quad x_2 - x_3 > 0 \quad (1)$$

$$g_2 \quad x_2 x_3 x_4 + 3e x_1 x_3 x_4 - 6k_1 h_t \ell_t^2 [x_1 + (x_2^2 - x_3^2)^{\frac{1}{2}}] > 0 \quad (2)$$

$$g_3 \quad x_2 x_3 x_5 + 3e x_1 x_3 x_5 - 6k_1 h_m \ell_m^2 [x_1 + (x_2^2 - x_3^2)^{\frac{1}{2}}] > 0 \quad (3)$$

$$g_4 \quad x_2 x_3 x_6 + 3e x_1 x_3 x_6 - 6k_1 h_b \ell_b^2 [x_1 + (x_2^2 - x_3^2)^{\frac{1}{2}}] \geq 0 \quad (4)$$

$$g_5 \quad x_2 x_3^2 x_4 + 3e x_1 x_3^2 x_4 - 26.4(k_1 h_t \ell_t^2)^{4/3} [x_1 + (x_2^2 - x_3^2)^{\frac{1}{2}}]^{4/3} > 0 \quad (5)$$

$$g_6 \quad x_2 x_3^2 x_5 + 3e x_1 x_3^2 x_5 - 26.4(k_1 h_m \ell_m^2)^{4/3} [x_1 + (x_2^2 - x_3^2)^{\frac{1}{2}}]^{4/3} > 0 \quad (6)$$

$$g_7 \quad x_2 x_3^2 x_6 + 3e x_1 x_3^2 x_6 - 26.4(k_1 h_b \ell_b^2)^{4/3} [x_1 + (x_2^2 - x_3^2)^{\frac{1}{2}}]^{4/3} \geq 0 \quad (7)$$

$$g_8 \quad x_4 - t_t^{\min} \geq 0 \quad (8)$$

$$g_9 \quad x_4 - [.39 \cdot 1.05(.01h_{1t})^{\frac{1}{2}}(.01x_1) + k_2] \geq 0 . \quad (9)$$

$$g_{10} \quad x_4 - [.39 \cdot 1.05(.01h_{1t})^{\frac{1}{2}}(.01x_2) + k_2] \geq 0 . \quad (10)$$

$$g_{11} \quad x_5 - t_m^{\min} \geq 0 . \quad (11)$$

$$g_{12} \quad x_5 - [.39 \cdot 1.05(.01h_{1m})^{\frac{1}{2}}(.01x_1) + k_2] \geq 0 . \quad (12)$$

$$g_{13} \quad x_5 - [.39 \cdot 1.05(.01h_{1m})^{\frac{1}{2}}(.01x_2) + k_2] \geq 0 . \quad (13)$$

$$g_{14} \quad x_6 - t_b^{\min} \geq 0 . \quad (14)$$

$$g_{15} \quad x_6 - [.39 \cdot 1.05(.01h_{1b})^{\frac{1}{2}}(.01x_1) + k_2] \geq 0 . \quad (15)$$

$$g_{16} \quad x_6 - [.39 \cdot 1.05(.01h_{1b})^{\frac{1}{2}}(.01x_2) + k_2] \geq 0 . \quad (16)$$

$$g_{17} - g_{18} \quad x_i \geq 0 ; i = 1 \text{ and } 3 . \quad (17-18)$$

5. COMPUTATIONAL RESULTS

Phase 1: Solution of the Problem $P(\bar{\epsilon})$

This phase involves the solution of the problem without parameter perturbation. Denote by ϵ the vector of design parameters listed in Table 1, i.e., the i th component of ϵ designating the i th parameter in Table 1. With this notation, let $\bar{\epsilon}$ represent the vector whose components are the respective data given in Table 1. Therefore, the design problem given in the previous section, with the problem design parameters equal to the values given in Table 1, is denoted $P(\bar{\epsilon})$, in conformance with the notation of Section 2.

Table 2 gives the calculated solution $x(\bar{\epsilon})$ (to two significant figures) of $P(\bar{\epsilon})$, along with the solution presented in [8]. Our optimal

TABLE 2
OPTIMAL SOLUTIONS FOR
PROBLEM P(ϵ)

Starting Point (cm)	Solution in [7] (cm)	Our Solution $x(\epsilon)$ (cm)
$x_1 = b_1 = 45.80$	57.80	57.82
$x_2 = b_2 = 43.20$	57.80	57.82
$x_3 = d = 30.50$	37.80	35.69
$x_4 = t_t = 1.20$	1.05	1.05
$x_5 = t_m = 1.20$	1.05	1.05
$x_6 = t_b = 1.30$	1.05	1.05
$f = w = 6.40$ tons	5.34 tons	5.25 tons

solution matches that of reference [8] except for x_3 (the corrugation depth). This slight difference in the value of x_3 , and consequently in the value of objective function W is believed to be due solely to the difference in manipulation of the input data to the model. In the current study the design parameter values were manipulated separately in the model, as encountered in a given calculation. The results given in [8] probably involved the calculation of coefficients (which are themselves functions of design parameters) "externally," using the aggregate results as input data to the model. The rounding of coefficients so calculated could readily explain the resulting minor discrepancy.

Phase 2: Sensitivity Analysis of Problem $P(\bar{\epsilon})$ for Right Hand Side Constraint Perturbations

The problem $P(\bar{\epsilon})$ with right hand side perturbation α of the constraints will be called Problem $R(\alpha)$. Obviously, $R(0) = P(\bar{\epsilon})$, so the optimal solution $x(\bar{\epsilon})$ of $P(\bar{\epsilon})$ given in Table 2 is also $x(0)$, the optimal solution of $R(0)$.

Denote by $\alpha = \bar{c}$ the vector α , all of whose components are equal to the number c . It was desired to analyze the solution of $R(\alpha)$ for $\alpha = \pm .25$ and $\alpha = \pm .50$.

The optimal solutions $x(\alpha)$ and the first order sensitivities $\nabla_{\alpha} f^*(\alpha)$, $\nabla_{\alpha} x(\alpha)$ and $\nabla_{\alpha} u(\alpha)$ were obtained for the given values of α . The following results were also obtained.

- (1) The assumptions of Lemma 2.1 were shown to hold at $x(0)$ and the results of the Lemma were computationally corroborated, for $\alpha = \pm .25$ and $\alpha = .50$.
- (2) Using the solution and first order sensitivity information associated with Problem $R(0)$, the optimal solution for the perturbed problems $R(\pm .25)$ and $R(\pm .50)$ were estimated and compared to those obtained by direct solution.

The following elaborates on the above results (1) and (2). Table 3 depicts the Lagrange multipliers, constraint values and sensitivity information associated with the optimal value function. It is clear from this table that $x(0)$ is feasible, the multipliers are nonnegative and complimentary slackness is satisfied (to two significant places).

TABLE 3

LAGRANGE MULTIPLIERS AND CONSTRAINT
VALUES FOR PROBLEM R(0) AT $x(0)$

(Excluding non-negativity constraints)

(i) Constraint No.	Value of Constraint g_i	Value of Lagrange Multiplier u_i	$\frac{\partial f^*(0)}{\partial x_i}$
1	22.13	$.4518784 \times 10^{-2}$	$.4518777 \times 10^{-2}$
2	2117.22	$.4723479 \times 10^{-4}$	$.4723564 \times 10^{-4}$
3	1385.41	$.7218857 \times 10^{-4}$	$.7218987 \times 10^{-4}$
4	1868.92	$.5351946 \times 10^{-4}$	$.535204 \times 10^{-4}$
5	41993.15	$.2381633 \times 10^{-5}$	$.2381606 \times 10^{-5}$
* 6	.043	$.2300013 \times 10^1$	$.2300013 \times 10^1$
7	27945.57	$.3580375 \times 10^{-5}$	$.3580337 \times 10^{-5}$
* 8	$.48 \times 10^{-5}$	$.2078305 \times 10^7$	$.2078305 \times 10^7$
9	.25	.3940568	.3940568
10	.25	.3940570	.3940570
* 11	$.96 \times 10^{-5}$	$.1040634 \times 10^7$	$.1040634 \times 10^7$
12	.10	.9604345	.9604345
13	.10	.9604360	.9604360
* 14	$.49 \times 10^{-4}$	$.2029215 \times 10^6$	$.2029215 \times 10^6$
* 15	$.31 \times 10^{-4}$	$.3221582 \times 10^6$	$.3221582 \times 10^6$
* 16	$.13 \times 10^{-4}$	$.7974950 \times 10^6$	$.7974950 \times 10^6$

Constraints marked by asterisk (*) are binding.

Based on Table 3 it was concluded that the constraints $g_6, g_8, g_{11}, g_{14}, g_{15}$ and g_{16} were binding. Rather large value of the constraint g_6 compared to the values of the remaining binding constraints may encourage the skeptical reader to disagree with the fact that this constraint is binding. However, one should notice that the coefficients involved in constraint g_6 are very large compared to the coefficients of the remaining binding constraints (this is also reflected in the values of the components of ∇g_6 listed in Table 4). By proper scaling of this constraint one may reduce its value to the order of the values of the remaining binding constraints. A sufficient reason that constraint g_6 may be concluded to be binding is that the optimal value function is sensitive to parameter k_1 (to be shown in Phase 3) while k_1 appears neither in the objective function nor in any of the binding constraints but constraint g_6 . Incidentally, this gives an interesting example of the sort of insight that is provided by a thorough sensitivity analysis.

Table 4 depicts the gradients of these binding constraints and their corresponding optimal Lagrange multipliers at the solution point for Problem $R(0)$. It is clear that the binding constraint gradients are linearly independent (A_3) and the associated Lagrange multipliers are positive. Therefore strict complementarity slackness (A_4) holds at $x(0)$. Since $x(0)$ was calculated to be a stationary point of the Lagrangian function, it follows from these facts that (A_2) is satisfied. This also means that $x(0)$ is indeed an isolated local solution. Finally the problem functions are all twice differentiable (A_1). Therefore, relative to two significant figures of accuracy, the four assumptions of the Lemma 2.1 have been verified for Problem $R(0)$. So the results of that lemma hold for Problem $R(\alpha)$, with α near zero.

The last column in Table 3 depicts the sensitivity of the optimal value function for Problem $R(0)$. As it is seen the entries in this column closely correspond to the respective Lagrange multipliers, listed in column 3 of Table 3, as expected from result (d) of Lemma 2.1.

TABLE 4

GRADIENTS OF THE BINDING CONSTRAINTS
AND CORRESPONDING LAGRANGE MULTIPLIERS
FOR PROBLEM $R(\alpha)$ AT $x(\alpha)$

	∇g_6	∇g_8	∇g_{11}	∇g_{14}	∇g_{15}	∇g_{16}
$R(-.25)$	-288	0	0	0	-.01557	0
	-3352	0	0	0	0	-.01557
	15752	0	0	0	0	0
	0	1	0	0	0	0
	313158	0	1	0	0	0
	0	0	0	1	1	1
$u(-.25)$	2.53	2.16×10^6	8.84×10^5	1.98×10^5	3.23×10^5	8.51×10^5
$R(0)$	-183	0	0	0	-.01557	0
	-2975	0	0	0	0	-.01557
	17393	0	0	0	0	0
	0	1	0	0	0	0
	250354	0	1	0	0	0
	0	0	0	1	1	1
$u(0)$	2.30	2.07×10^6	1.04×10^6	2.03×10^5	3.22×10^5	7.97×10^5

TABLE 4 (Cont'd)
 GRADIENTS OF THE BINDING CONSTRAINTS
 AND CORRESPONDING LAGRANGE MULTIPLIERS
 FOR PROBLEM $R(\alpha)$ AT $x(\alpha)$

	∇g_6	∇g_8	∇g_{11}	∇g_{14}	∇g_{15}	∇g_{16}
$R(\overline{.25})$	-108	0	0	0	-.01557	0
	-2752	0	0	0	0	-.01557
	18962	0	0	0	0	0
	0	1	0	0	0	0
	208223	0	1	0	0	0
	0	0	0	1	1	1
$u(\overline{.25})$	2.17	2.03×10^6	1.13×10^6	2.06×10^5	3.21×10^5	7.67×10^5
$R(\overline{.50})$	-58	0	0	0	-.01557	0
	-2839	0	0	0	0	-.01557
	20438	0	0	0	0	0
	0	1	0	0	0	0
	178125	0	1	0	0	0
	0	0	0	1	1	1
$u(\overline{.50})$	2.08	2.003×10^6	1.19×10^6	2.08×10^5	3.2×10^5	7.47×10^5

Table 5 depicts the optimal solutions, optimal values, and a typical Lagrange multiplier $u_g(\alpha)$ for problems $R(\alpha)$, where $\alpha = \pm .25, \pm .50$, obtained by direct calculation. Figures 4, 5 and 6 suggest that $x(\alpha)$, $f^*(\alpha)$ and $u_g(\alpha)$ are continuous and smooth functions of α for the entire range of parameters considered (a result known to hold for α near 0, by conclusion (b) of Lemma 2.1), where $\alpha = \bar{c} = c(1,1,\dots,1)^T$.

Table 6 indicates the binding constraints (using two significant figures) for Problem $R(\alpha)$ at $x(\alpha)$. As may be seen, the binding constraints for all perturbed problems, except $R(-.50)$, remain the same. Moreover, Table 4 shows that the gradients of these binding constraints are linearly independent and we determined that strict complementarity slackness holds true. So, the results of Lemma 2.1 must also hold for the problems $R(+.25)$ and $R(+.50)$. The change in the binding constraint index set for problem $R(-.50)$ means that the perturbation $\alpha = -.50$ is not small enough to retain the general solution structure of the problem.

In order to estimate the optimal solution of Problem $R(\alpha)$ based on information from Problem $R(0)$ the following first order (Taylor's Series) extrapolation formula was used,

$$y(\alpha) = y(0) + V_{\alpha}^T y(0) \cdot \alpha$$

where $y(0)$ and $V_{\alpha} y(0)$ are the estimates of a Kuhn-Tucker triple and its derivatives with respect to α for problem $R(0)$. Although the magnitude of the perturbations were quite significant, the extrapolated values are in close agreement with those of direct calculation. Extrapolation was also done for Problem $R(-.50)$ to study the discrepancy between extrapolated and calculated results. Table 7 shows these results and Figure 7 depicts some of these, namely $x_3(\alpha)$ and $x_5(\alpha)$, graphically.

In order to computationally check the stability of the algorithm's estimates of solution and sensitivity information of Problem $R(\alpha)$, $f(\cdot)$, $x(\cdot)$, $u(\cdot)$ and their partial derivatives with respect to α were recorded along the algorithm solution trajectory. Values along the trajectory path are plotted in Figure 8 for a sample of typical values, for the Problem $R(.50)$. As it is seen, the solution estimates are virtually constant from the third subproblem onwards.

TABLE 5

OPTIMAL SOLUTIONS FOR
PROBLEM $R(\alpha)$

	$R(-.50)$	$R(-.25)$	$R(0)$	$R(.25)$	$R(.50)$
x_1	58.45	57.82	57.82	57.82	57.82
x_2	58.45	57.82	57.82	57.82	57.82
x_3	45.50	39.91	35.69	32.54	30.10
x_4	.55	.8	1.05	1.30	1.55
x_5	.61	.8	1.05	1.30	1.55
x_6	.56	.8	1.05	1.30	1.55
u_8	2282964	2156015	2067079	2033079	2003181
$f^*(\alpha)$	3.14	4.15	5.25	6.36	7.47

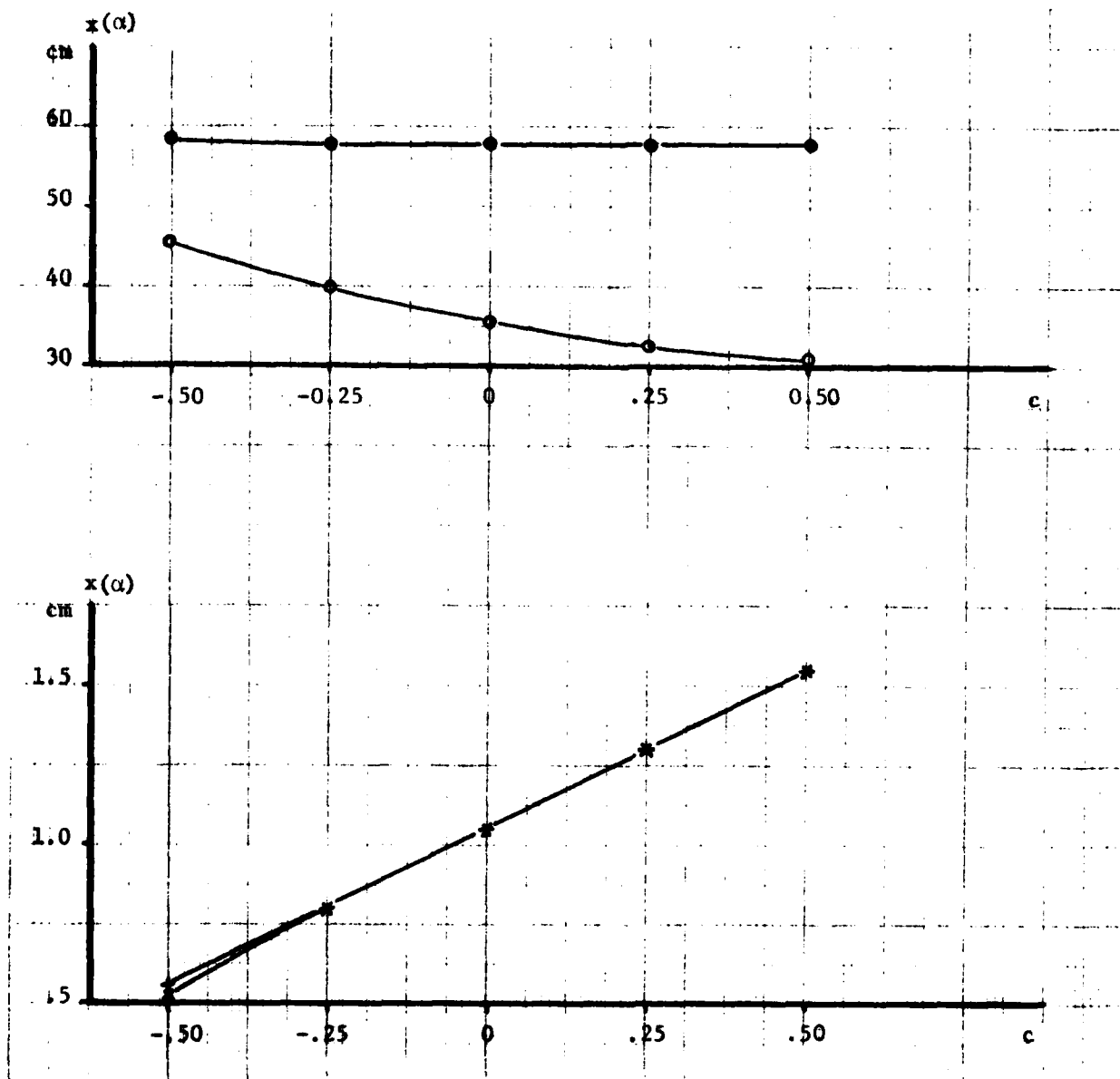


Figure 4. $x(\alpha)$ versus c for Problem $R(\alpha)$, where $\alpha = \bar{c}$.

LEGENDS

● for x_1 and x_2

○ for x_3

x for x_4 and x_5 Common

+ for x_6

* for x_4, x_5 and x_6 Common

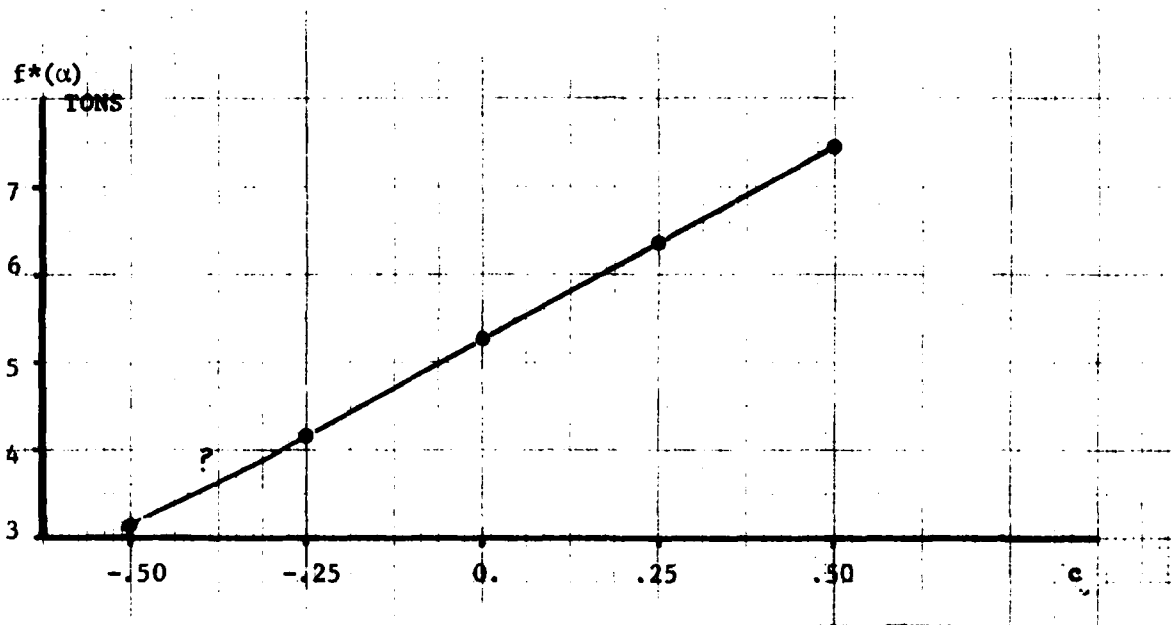


Figure 5. $f^*(\alpha)$ versus c for Problem $R(\alpha)$, where $\alpha = \bar{c}$.

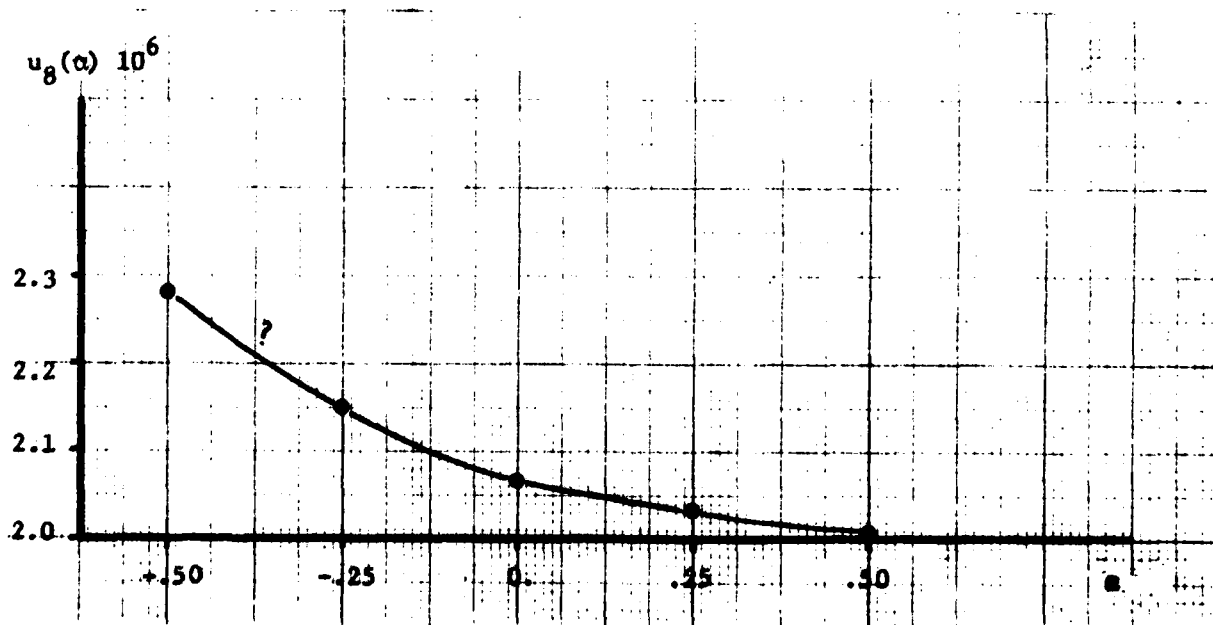


Figure 6. $u_8(\alpha)$ versus c for Problem $R(\alpha)$, where $\alpha = \bar{c}$.

TABLE 6
 BINDING CONSTRAINTS FOR PROBLEM
 $R(\alpha)$ AT $x(\alpha)$

α Constraint	$-.50$	$-.25$	0	$+.25$	$+.50$
g_1	-	-	-	-	-
g_2	-	-	-	-	-
g_3	B	-	-	-	-
g_4	B	-	-	-	-
g_5	-	-	-	-	-
g_6	-	B	B	B	B
g_7	-	-	-	-	-
g_8	B	B	B	B	B
g_9	-	-	-	-	-
g_{10}	-	-	-	-	-
g_{11}	-	B	B	B	B
g_{12}	-	-	-	-	-
g_{13}	-	-	-	-	-
g_{14}	-	B	B	B	B
g_{15}	B	B	B	B	B
g_{16}	B	B	B	B	B

where

B stands for binding constraints
 - stands for nonbinding constraints

TABLE 7
EXTRAPOLATED SOLUTIONS VERSUS CALCULATED SOLUTIONS
FOR PROBLEM $R(\alpha)$

		$R(-.50)$	$R(-.25)$	$R(0)$	$R(.25)$	$R(.50)$
x_1	e^1	57.82	57.82	-	57.82	57.82
	c^2	58.45	57.82	57.82	57.82	57.82
x_2	e	57.82	57.82	-	57.82	57.82
	c	58.45	57.82	57.82	57.82	57.82
x_3	e	42.88	39.28	-	32.09	28.49
	c	45.50	39.91	35.69	32.54	30.10
x_4	e	.55	.80	-	1.30	1.55
	c	.55	.80	1.05	1.30	1.55
x_5	e	.55	.80	-	1.30	1.55
	c	.61	.80	1.05	1.30	1.55
x_6	e	.55	.80	-	1.30	1.55
	c	.56	.80	1.05	1.30	1.55
f^*	e	3.027	4.137	-	6.358	7.469
	c	3.136	4.145	5.248	6.356	7.467

1_e , extrapolated

2_c , calculated

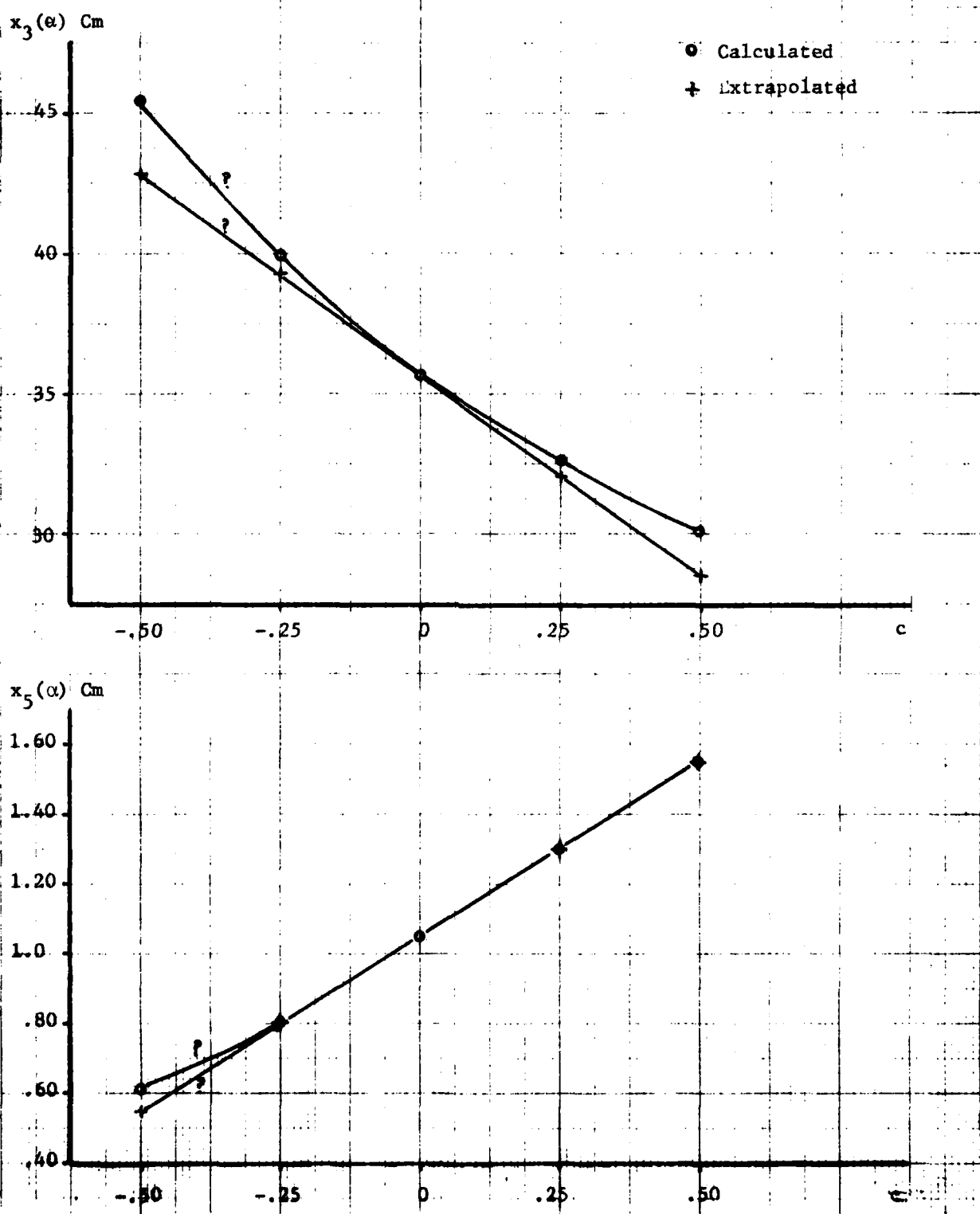


Figure 7. $x_3(\alpha)$ and $x_5(\alpha)$ versus c for Problem $R(\alpha)$, where $\alpha = \bar{c}$.
(Extrapolated compared to calculated).

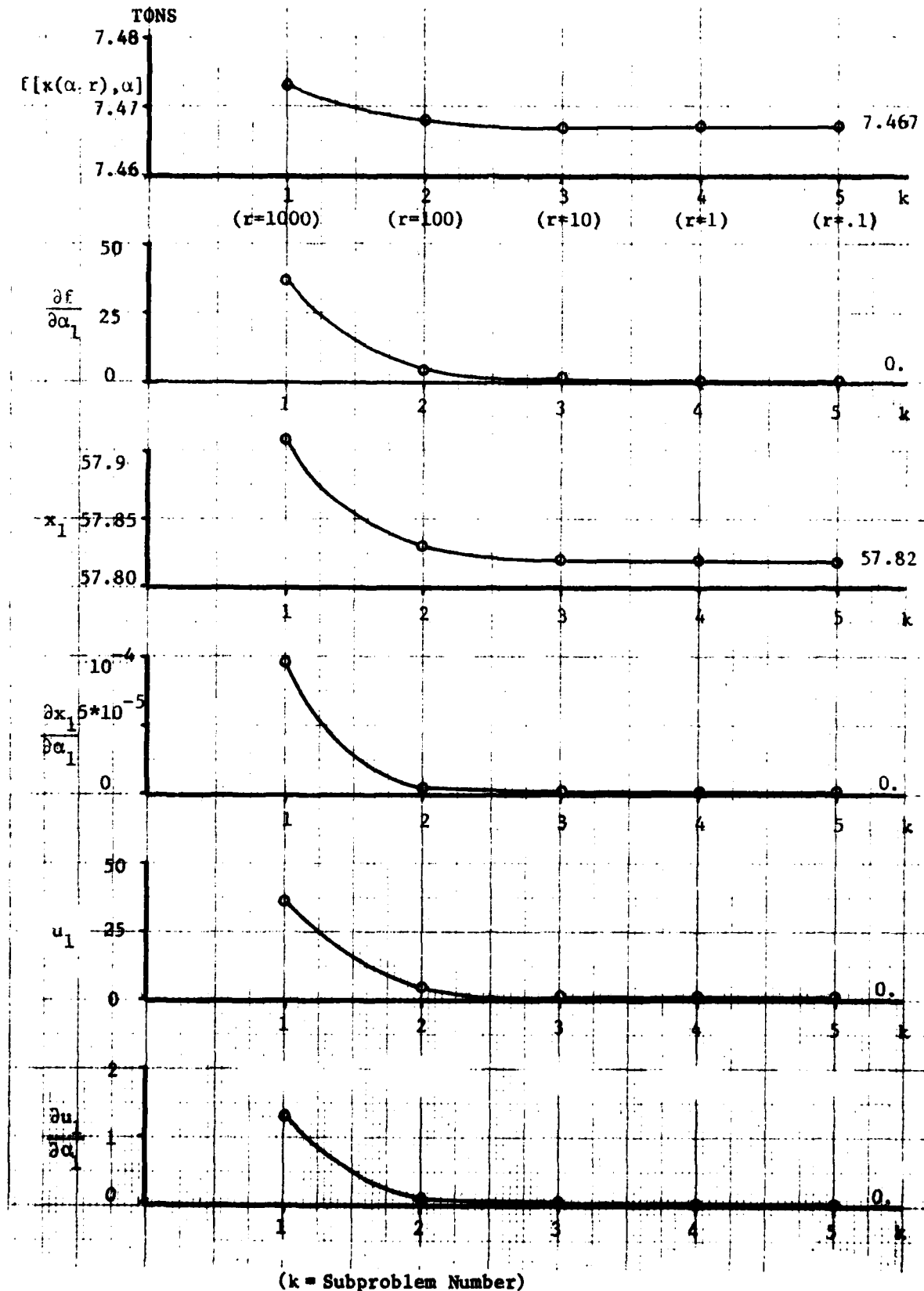


Figure 8. Sample of subproblem solution point and partial derivatives along the trajectory for Problem R(.50).

Numerous inferences can be drawn from such information. For example, note that the constraints g_8 , g_{11} and g_{14} which correspond to the minimum plate thickness requirement for top, middle, and bottom panels remain binding for the perturbed problems over the range of perturbations considered (i.e., $-.25$ to $+.50$). On the other hand, because the weight of the bulkhead is linearly proportional to the plate thickness, any relaxation on minimum thickness requirement would decrease the bulkhead weight (objective function) by a considerable amount. Based on the results given in Table 3, the saving on bulkhead material weight will be about

$$.207 + .104 + .020 = .331 \text{ tons}$$

per one millimeter relaxation of the required plate thickness. Figure 5, which shows the optimal function values versus perturbations, confirms this observation.

Phase 3: Sensitivity Analysis with Respect to Design Parameter Perturbations

This study phase seeks to obtain sensitivity information when the design parameters are perturbed. As noted, Table 1 lists all the design parameters and their initial values.

The parameters ρ (density of steel), e (dimensionless constant in section modulus formula), B (the width of the panel) and h_a (the distance from top of the bulkhead to free liquid level) are assumed to be subject to negligible variation and so their perturbations are excluded from the sensitivity analysis.

It is important to note that the design parameters ℓ_t , ℓ_m , ℓ_b , h_t , h_m , h_b , h_{1t} , h_{1m} and h_{1b} are not independent. For example, as can easily be verified from Figure 3, these parameters may be expressed in terms of ℓ_t , ℓ_m and h_{1b} as follows:

$$\begin{aligned}\ell_b &= h_{1b} - (250 + \ell_t + \ell_m) \\ h_{1t} &= 250 + \frac{\ell_t}{2} \\ h_m &= 250 + \ell_t + \frac{\ell_m}{2} \\ h_b &= 125 + \frac{\ell_t + \ell_m + h_{1b}}{2}\end{aligned}$$

$$h_{1t} = 250 + \ell_t$$

$$h_{1m} = 250 + \ell_t + \ell_m$$

In this analysis of the problem, we explicitly impose these equations and express all the design parameters as functions of ℓ_t , ℓ_m and h_{1b} , as indicated. This eliminates unnecessary sensitivity calculations and strictly accommodates the dependencies.

The "independent parameters" on which the sensitivity is analyzed are shown in Table 8. For convenience, we relabel the j th parameter ϵ_j .

As previously noted, the corresponding numerical values of ϵ_j are given in Table 1. These values are taken as the initial value of the problem parameters and are again designated by $\bar{\epsilon}$.

Aside from an analysis of the Problem $P(\bar{\epsilon})$, the problem was also analyzed under six different perturbations. Namely, all of the independent design parameters were perturbed simultaneously by the same fraction of their original values. The perturbations were +1, +2 and +5 percent of the original problem parameter vector $\bar{\epsilon}$. The corresponding problems are hence designated by $P(1.01\bar{\epsilon})$, $P(.99\bar{\epsilon})$, $P(1.102\bar{\epsilon})$, $P(.98\bar{\epsilon})$, $P(1.05\bar{\epsilon})$ and $P(.95\bar{\epsilon})$, respectively, following our usual notation.

After calculating the optimal solutions and corresponding first order sensitivity information for the perturbed problems, the following analysis was completed:

- (1) For small perturbations the results of Lemma 2.1 for Problem $P(\bar{\epsilon})$ were verified,
- (2) Using the solution and the first order sensitivity information of the Problem $P(\bar{\epsilon})$, optimal solutions for the Problem $P(\epsilon)$ were estimated and compared with those obtained by direct solution.

Tables 8, 9, and 10 show the sensitivities of the optimal value function, solution point and Lagrange multipliers with respect to the independent design

TABLE 8
 SENSITIVITY OF $f^*(\bar{\epsilon})$ WITH RESPECT TO THE INDEPENDENT
 DESIGN PARAMETERS ϵ_j FOR PROBLEM P($\bar{\epsilon}$)

Parameter Index j	Parameter ϵ_j	$\frac{\partial f^*(\bar{\epsilon})}{\partial \epsilon_j}$
1	ℓ_t	4393
2	ℓ_m	8582
3	h_{tb}	349
4	t_t^{\min}	2078305
5	t_m^{\min}	1040634
6	t_b^{\min}	202921
7	k_2	1119656
8	k_1	1161×10^{10}

TABLE 9
SOLUTION POINT SENSITIVITY $\partial x_1 / \partial \epsilon_j$
FOR PROBLEM P($\bar{\epsilon}$)

	i=1	2	3	4	5	6
j=1	$.2 \times 10^{-5}$	$.2 \times 10^{-5}$	- *	-	-	-
2	-.015842	-.013829	.10226	-	-	$-.208 \times 10^{-3}$
3	-.020053	-.020053	$-.3625 \times 10^{-2}$	-	-	-
4	$.774 \times 10^{-3}$	726×10^{-3}	$.119 \times 10^{-3}$	1.	-	$.11 \times 10^{-4}$
5	$-.1297 \times 10^{-2}$	$-.123 \times 10^{-2}$	-14.39778	-	.999999	$-.18 \times 10^{-4}$
6	64.2357	64.23601	11.61322	-	-	.999925
7	-64.23520	-64.23554	-11.61314	-	-	$.82 \times 10^{-4}$
8	$-.4453 \times 10^{+8}$	$-.3887 \times 10^{+8}$	$.2874 \times 10^{+9}$	-1.14863	-1831.179	$-.5848 \times 10^{+6}$

*Sensitivities with less than 10^{-6} in absolute value are indicated by
(-) in the above table.

TABLE 10

LAGRANGE MULTIPLIERS SENSITIVITY $\partial u_i / \partial \epsilon_j$
FOR PROBLEM P($\bar{\epsilon}$)

$i=1$	2	3	4	5	6	7	8
1	-	$.3 \times 10^{-5}$	-	$-.2939 \times 10^{-2}$	$-.010746$	$.010746$	66646.
2	-	-	$-.156 \times 10^{-3}$	$.79 \times 10^{-4}$	$-.81 \times 10^{-4}$	$.81 \times 10^{-4}$	123.6
3	-	-	-	$-.176 \times 10^{-3}$	$-.139 \times 10^{-3}$	$.138 \times 10^{-3}$	720.9
4	-	-	-	$.102 \times 10^{-3}$	$-.296 \times 10^{-3}$	94×10^{-4}	357.
5	-	-	$-.14 \times 10^{-4}$	$.14 \times 10^{-4}$	$-.4 \times 10^{-5}$	$.4 \times 10^{-5}$	-40.1
6	$.2035 \times 10^{-2}$	-14.71785	.9589	3891.7	47145.7	-47145.	$-.384 \times 10^{13}$
7	-	-	-	32×10^{-4}	$-.38 \times 10^{-4}$	$.6 \times 10^{-5}$	-44.58
8	4182.802	137.6201	$-.1854 \times 10^{+9}$	-226370.	-440833.6	440828.7	$.491 \times 10^{13}$
9	-	$-.348 \times 10^{-3}$	-1.5528	$-.2 \times 10^{-5}$	1.1149	.4378	-774423.6
10	-	$-.348 \times 10^{-3}$	-1.5528	$-.2 \times 10^{-5}$	1.1149	.4378	-676199.7
11	-480.4356	-274.9426	-226371.	$-.107 \times 10^8$	736654.	-880732.5	$.199 \times 10^{17}$
12	-	$-.2546 \times 10^{-2}$	$.9 \times 10^{-5}$	-9.2244	8.1571	1.0672	-5648779.
13	-	$-.2546 \times 10^{-2}$	$.9 \times 10^{-5}$	-9.2244	8.1571	1.06721	-4930175.
14	-935.2219	-656.6095	-440835.	736655.	3050630.	-3346975.	$.241 \times 10^{18}$
15	269.4467	189.6840	126985.	-212956.	-885330.6	958413.5	$-.112 \times 10^{18}$
16	666.7745	553.7159	313850.	-667759.	-2461661.	-1383566.	$-.129 \times 10^{18}$

Sensitivities with less than 10^{-6} in absolute value are indicated by (-) in the above table.

parameters of Problem P(c). As shown in Table 8 the optimal value function is extremely sensitive to minimum allowable panel thicknesses t_t^{\min} , t_m^{\min} , t_b^{\min} and corrosion allowance k_2 and apparently sensitive to the allowable bending stress factor k_1 . It is only partially sensitive to panel lengths ℓ_t and ℓ_m , and practically insensitive to the total depth of panel h_{lb} .

It is well-known that in order to obtain a meaningful interpretation of the implications of a sensitivity analysis, the information must be carefully analyzed in the perspective of the application. For example, the parameter derivatives of the objective function $f^*(0)$ are the instantaneous rates of change. A rate of change can itself change significantly away from the base point, thus rendering extrapolations invalid for finite parameter changes. Thus, further analyses such as those reported here are essential. Another caution must be directed to equating the rate of change with a change "due to a one unit change" in the given parameter value. It is a change per unit change, but only at a given parameter value.

A unit change in a parameter value may not be feasible or meaningful. For example, the initial value of the parameter k_1 is $.694 \times 10^{-7} \text{ cm}^{-1}$. A one unit decrease is not possible. A "reasonable" change for this parameter might be expected to be on the order of $\pm 10^{-9}$. Given this scaling, it follows that the objective function is only relatively mildly sensitive to the changes in this parameter. Similarly, the sensitivity of the parameters k_2 and the minimum allowable panel thicknesses should all probably be scaled by a factor of 10^{-1} . Appropriate scaling of parameter changes depends on the context of the application, parameter interactions that may not even be explicitly represented in the model and other considerations. This is ultimately a "management" decision and is beyond the scope of this paper. We shall assume that specified feasible changes are stipulated and concentrate on measuring their effects.

Table II and Figure 9 show the calculated optimal solutions for Problem P(c), when it is stipulated that the given parameter values be altered by the indicated fractions of their original values.

TABLE 11
OPTIMAL SOLUTIONS FOR
PROBLEM P (ϵ)

	$P(0.95\bar{\epsilon})$	$P(0.98\bar{\epsilon})$	$P(0.99\bar{\epsilon})$	$P(\bar{\epsilon})$	$P(1.01\bar{\epsilon})$	$P(1.02\bar{\epsilon})$	$P(1.05\bar{\epsilon})$
x_1 (cm)	63.02	61.53	57.25	57.82	58.42	59.01	61.64
x_2 (cm)	63.04	61.53	57.26	57.82	58.43	59.02	61.65
x_3 (cm)	38.93	35.90	35.19	35.69	36.2	36.71	38.41
x_4 (cm)	1.0	1.03	1.04	1.05	1.06	1.07	1.10
x_5 (cm)	1.0	1.03	1.04	1.05	1.06	1.07	1.10
x_6 (cm)	1.13	1.10	1.04	1.05	1.06	1.07	1.12
$u_8(\epsilon)$	1964924	2009085	2055230	2078305	2101457	2125542	2189718
$f^*(\epsilon)$ (Tons)	5.176	5.175	5.190	5.248	5.306	5.365	5.541

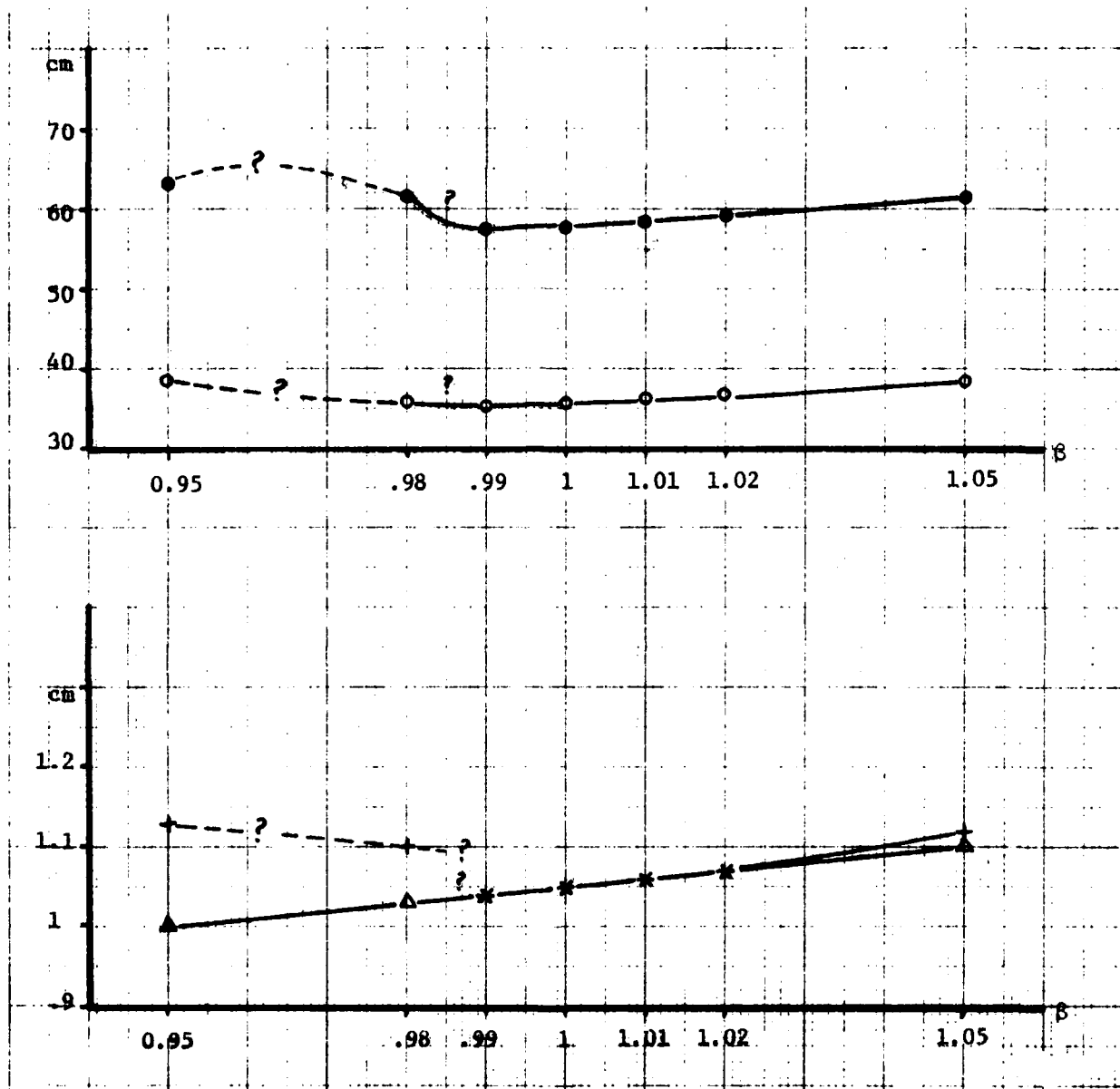


Figure 9. $x(\epsilon)$ versus β . for Problem $P(\epsilon)$, where $\epsilon = \beta\bar{\epsilon}$.

LEGENDS ● for x_1 and x_2 ▲ for x_4 and x_5 Common
 ● for x_3 + for x_6
 * for x_4, x_5 and x_6 Common

It is obvious that the assumptions of the main sensitivity theorem holds true for $P(\bar{\epsilon})$ (because $P(\bar{\epsilon})$ is identical to $R(0)$ and we have already verified the validity of these assumptions for $R(0)$ in the previous part). So the results of this theorem should also hold true for Problem $P(\epsilon)$ provided that the introduced perturbations are not too large.

The result a of Lemma 2.1 is immediate. Figures 9 and 10 are included to illustrate the result b of this lemma. As depicted, $x(\epsilon)$ and $u_8(\epsilon)$ (a sample of optimal Lagrange multiplier) appear to be smooth functions of β , for β between 0.99 and 1.05, where $\epsilon = \beta\bar{\epsilon}$. However there is a definite sharp change (or discontinuity) for $u_8(\epsilon)$ as well as $x_1(\epsilon)$, $x_2(\epsilon)$ and $x_6(\epsilon)$ for some value of ϵ between $0.99\bar{\epsilon}$ and $0.98\bar{\epsilon}$. This may be due to the fact that perturbations more than $-0.01\bar{\epsilon}$ (in magnitude) are not sufficiently small to keep the structure of the perturbed problem essentially unchanged. Examining the binding constraints for the perturbed problems confirms this. Table 12 shows the binding constraints of Problem $P(\epsilon)$ at different perturbations.

As shown for perturbations below $-0.01\bar{\epsilon}$. The binding constraints g_6 and g_{14} are well satisfied and constraints g_7 , g_{12} and g_{13} become binding. Moreover constraint g_{14} is no longer binding for perturbations of $0.05\bar{\epsilon}$. Furthermore, as depicted in Table 13, the gradients of binding constraints of Problem $P(\epsilon)$ are linearly independent and strict complementarity slackness holds true for ϵ between $0.99\bar{\epsilon}$ and $1.02\bar{\epsilon}$. (Result c of Lemma 2.1). So all the results of Lemma 2.1 for above range of perturbation (i.e., $0.99\bar{\epsilon}$ to $1.02\bar{\epsilon}$) are computationally verified.

As before estimates were made for the optimal solution of the perturbed Problem $P(\epsilon)$, based on the solution and sensitivity information of the unperturbed Problem $P(\bar{\epsilon})$, by first order extrapolation. Table 14 and Figure 11 show the results of this extrapolation and compare them with the results of direct calculation. It is obvious that one should not attempt to extrapolate from the solution of $P(\bar{\epsilon})$ to solutions of $P(0.98\bar{\epsilon})$, $P(0.95\bar{\epsilon})$ and $P(1.05\bar{\epsilon})$. We have deliberately calculated this extrapolation to show the consequences.

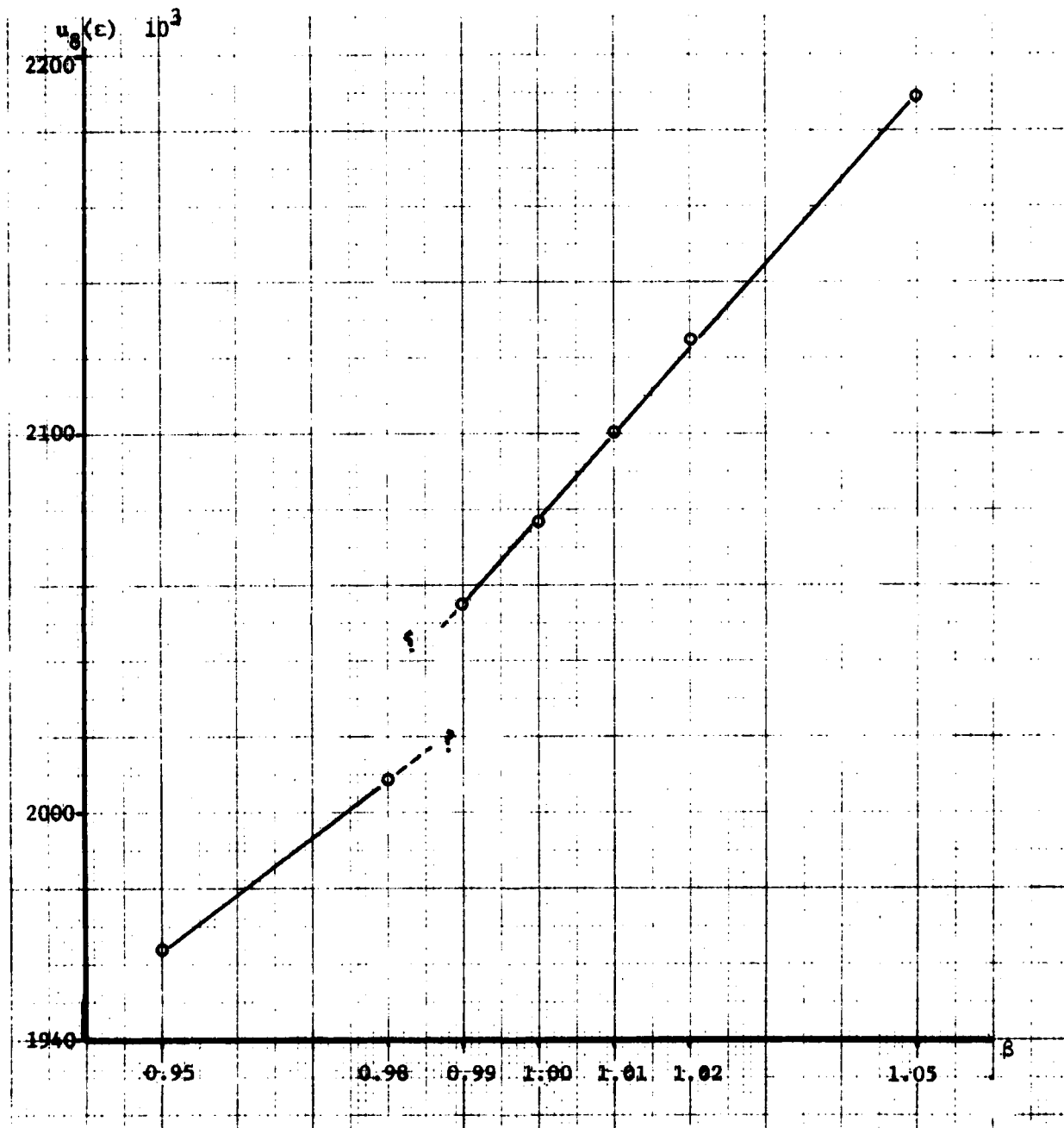


Figure 10. $u_8(\epsilon)$ versus β for Problem $P(\epsilon)$, where $c = \beta \bar{c}$.

TABLE 12

BINDING CONSTRAINTS FOR PROBLEM
 $P(\epsilon)$ AT $x(i)$

Constraint \ ϵ	0.95 $\bar{\epsilon}$	0.98 $\bar{\epsilon}$.99 $\bar{\epsilon}$	$\bar{\epsilon}$	1.01 $\bar{\epsilon}$	1.02 $\bar{\epsilon}$	1.05 $\bar{\epsilon}$
g_1	-	-	-	-	-	-	-
g_2	-	-	-	-	-	-	-
g_3	-	-	-	-	-	-	-
g_4	-	-	-	-	-	-	-
g_5	-	-	-	-	-	-	-
g_6	-	-	B	B	B	B	B
g_7	B	B	-	-	-	-	-
g_8	B	B	B	B	B	B	B
g_9	-	-	-	-	-	-	-
g_{10}	-	-	-	-	-	-	-
g_{11}	B	B	B	B	B	B	B
g_{12}	B	-	-	-	-	-	-
g_{13}	B	-	-	-	-	-	-
g_{14}	-	-	B	B	B	B	B
g_{15}	B	B	B	B	B	B	B
g_{16}	B	B	B	B	B	B	B

where

B stands for binding constraints

- stands for unbinding constraints.

TABLE 13

GRADIENTS OF THE BINDING CONSTRAINTS
AND CORRESPONDING LAGRANGE MULTIPLIERS
FOR PROBLEM $P(\epsilon)$ AT $x(\epsilon)$

	∇g_6	∇g_8	∇g_{11}	∇g_{14}	∇g_{15}	∇g_{16}
$P(0.99\bar{\epsilon})$	-173	0	0	0	-.01557	0
	-2848	0	0	0	0	-.01557
	16777	0	0	0	0	0
	0	1	0	0	0	0
	240914	0	1	0	0	0
	0	0	0	1	1	1
$u(0.99\bar{\epsilon})$	2.36	2.05×10^6	1.03×10^6	2.51×10^5	3.19×10^5	7.88×10^5
$P(\bar{\epsilon})$	-183	0	0	0	-.01557	0
	-2975	0	0	0	0	-.01557
	17393	0	0	0	0	0
	0	1	0	0	0	0
	250354	0	1	0	0	0
	0	0	0	1	1	1
$u(\bar{\epsilon})$	2.3	2.8×10^6	1.04×10^6	2.03×10^5	3.22×10^5	7.97×10^5

TABLE 13 (Cont'd)
 GRADIENTS OF THE BINDING CONSTRAINTS
 AND CORRESPONDING LAGRANGE MULTIPLIERS
 FOR PROBLEM $P(\epsilon)$ AT $x(\epsilon)$

	∇g_6	∇g_8	∇g_{11}	∇g_{14}	∇g_{15}	∇g_{16}
$P(1.01\bar{\epsilon})$	-194	0	0	0	-.01557	0
	-31070	0	0	0	0	-.01557
	18028	0	0	0	0	0
	0	1	0	0	0	0
	260078	0	1	0	0	0
	0	0	0	1	1	1
$u(1.01\bar{\epsilon})$	2.24	2.1×10^6	1.05×10^6	1.55×10^5	3.25×10^5	8.07×10^5
$P(1.02\bar{\epsilon})$	-206	0	0	0	-.01557	0
	-3244	0	0	0	0	-.01557
	18679	0	0	0	0	0
	0	1	0	0	0	0
	270062	0	1	0	0	0
	0	0	0	1	1	1
$u(1.02\bar{\epsilon})$	2.18	2.12×10^6	1.06×10^6	1.06×10^5	3.29×10^5	8.17×10^5

TABLE 14
EXTRAPOLATED SOLUTIONS VERSUS CALCULATED
SOLUTIONS¹ FOR PROBLEM P(ϵ)

	P(0.95 $\bar{\epsilon}$)	P(0.98 $\bar{\epsilon}$)	P(0.99 $\bar{\epsilon}$)	P($\bar{\epsilon}$)	P(1.01 $\bar{\epsilon}$)	P(1.02 $\bar{\epsilon}$)	P(1.05 $\bar{\epsilon}$)
$x_1 \frac{e^2}{c^3}$	<u>55.39</u>	<u>56.84</u>	<u>57.33</u>	-	<u>58.30</u>	<u>58.79</u>	<u>60.25</u>
c	63.29	61.54	57.24	57.82	58.39	58.97	61.52
$x_2 \frac{e}{c}$	<u>55.33</u>	<u>56.82</u>	<u>57.32</u>	-	<u>58.31</u>	<u>58.81</u>	<u>60.30</u>
c	63.29	61.54	57.24	57.82	58.39	58.97	61.52
$x_3 \frac{e}{c}$	<u>32.95</u>	<u>34.59</u>	<u>35.14</u>	-	<u>36.23</u>	<u>36.78</u>	<u>38.42</u>
c	38.92	35.90	35.18	35.69	36.19	36.70	38.39
$x_4 \frac{e}{c}$	<u>1.0</u>	<u>1.03</u>	<u>1.04</u>	-	<u>1.06</u>	<u>1.07</u>	<u>1.10</u>
c	1.0	1.03	1.04	1.05	1.06	1.07	1.10
$x_5 \frac{e}{c}$	<u>1.0</u>	<u>1.03</u>	<u>1.04</u>	-	<u>1.06</u>	<u>1.07</u>	<u>1.10</u>
c	1.0	1.03	1.04	1.05	1.06	1.07	1.10
$x_6 \frac{e}{c}$	<u>1.0</u>	<u>1.03</u>	<u>1.04</u>	-	<u>1.06</u>	<u>1.07</u>	<u>1.10</u>
c	1.13	1.12	1.04	1.05	1.06	1.07	1.11

¹Entries have been rounded to the nearest two decimal points.

²e, extrapolated

³c, calculated

As suggested by Figure 11, the extrapolated results for Problems $P(0.99\bar{\epsilon})$ and $P(1.02\bar{\epsilon})$ are virtually the same as those obtained by direct calculation, while the results for Problems $P(0.95\bar{\epsilon})$, $P(0.98\bar{\epsilon})$ and $P(1.05\bar{\epsilon})$ are quite different. For example, linear extrapolation from $P(\bar{\epsilon})$ to $P(0.95\bar{\epsilon})$ underestimates variable x_3 by about 6 cm, equivalent to about 15.3 percent of its correct value. (See Table 14).

It was also attempted to estimate the penalty function solution "trajectory" $x(\epsilon, r)$ for the perturbed Problem $P(\epsilon)$, using the solution point and corresponding sensitivity information along the solution trajectory of Problem $P(\bar{\epsilon})$. As shown in Table 15 for subproblem 2, with penalty function parameter $r=100$, the results of this extrapolation for ϵ between $.99\bar{\epsilon}$ and $1.02\bar{\epsilon}$ are quite satisfactory.

Figure 12 indicates the stability of $f^*(\epsilon)$, $x_1(\epsilon)$, $u_1(\epsilon)$ and their partial derivatives with respect to the parameter ϵ_1 along the solution trajectory of Problem $P(1.05\bar{\epsilon})$. As seen, the values for these quantities are again fairly stabilized from the third subproblem onwards.

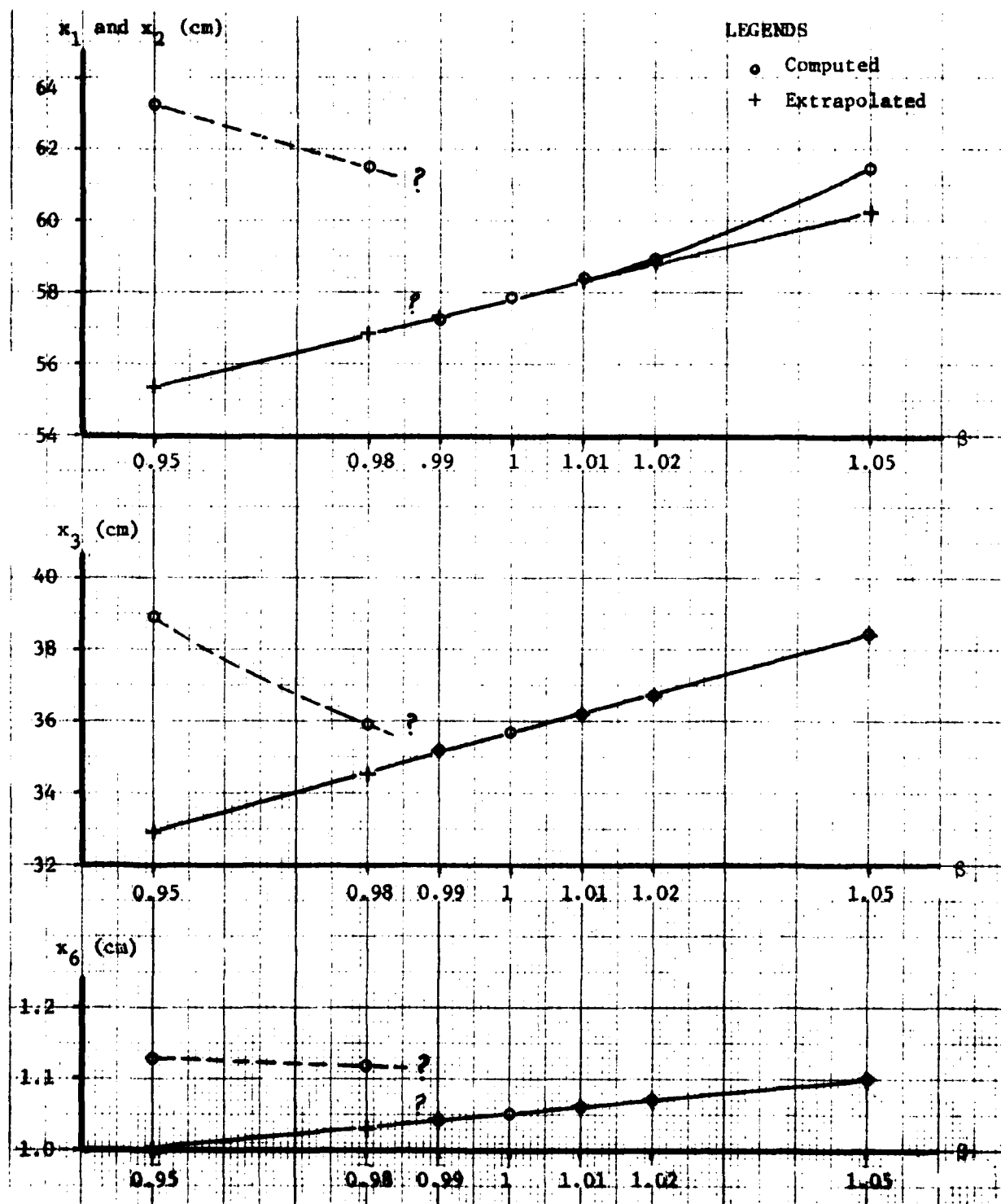


Figure 11. $x_1(\epsilon)$, $x_2(\epsilon)$, $x_3(\epsilon)$ and $x_6(\epsilon)$ versus β

for Problem P(ϵ), where $\epsilon = \beta\bar{\epsilon}$ (extrapolated compared to calculated).

TABLE 15

EXTRAPOLATED SOLUTIONS $x(\bar{\epsilon}, r)$ VERSUS CALCULATED
SOLUTIONS¹ FOR SUBPROBLEM 2² OF PROBLEM P(ϵ)

	P(0.95 $\bar{\epsilon}$)	P(0.98 $\bar{\epsilon}$)	P(0.99 $\bar{\epsilon}$)	P($\bar{\epsilon}$)	P(1.01 $\bar{\epsilon}$)	P(1.02 $\bar{\epsilon}$)	P(1.05 $\bar{\epsilon}$)
x_1 $\frac{e^3}{c^4}$	55.37	56.84	57.34	-	58.32	58.81	60.29
	63.02	61.53	57.25	57.82	58.42	59.01	61.64
x_2 $\frac{e}{c}$	55.33	56.83	57.34	-	58.34	58.85	60.36
	63.04	61.53	57.26	57.82	58.43	59.02	61.65
x_3 $\frac{e}{c}$	32.95	34.60	35.14	-	36.24	36.79	38.43
	38.93	35.90	35.19	35.69	36.20	36.71	38.41
x_4 $\frac{e}{c}$	1.0	1.02	1.03	-	1.06	1.08	1.12
	1.0	1.03	1.04	1.05	1.06	1.07	1.10
x_5 $\frac{e}{c}$.99	1.02	1.04	-	1.06	1.07	1.11
	1.0	1.03	1.04	1.05	1.06	1.07	1.10
x_6 $\frac{e}{c}$	1.0	1.03	1.04	-	1.06	1.07	1.10
	1.13	1.10	1.04	1.05	1.06	1.07	1.12

¹All entries are rounded to the nearest two decimal points.

²The penalty function parameter $r = 100$ in subproblem 2.

³e, extrapolated

⁴c, calculated

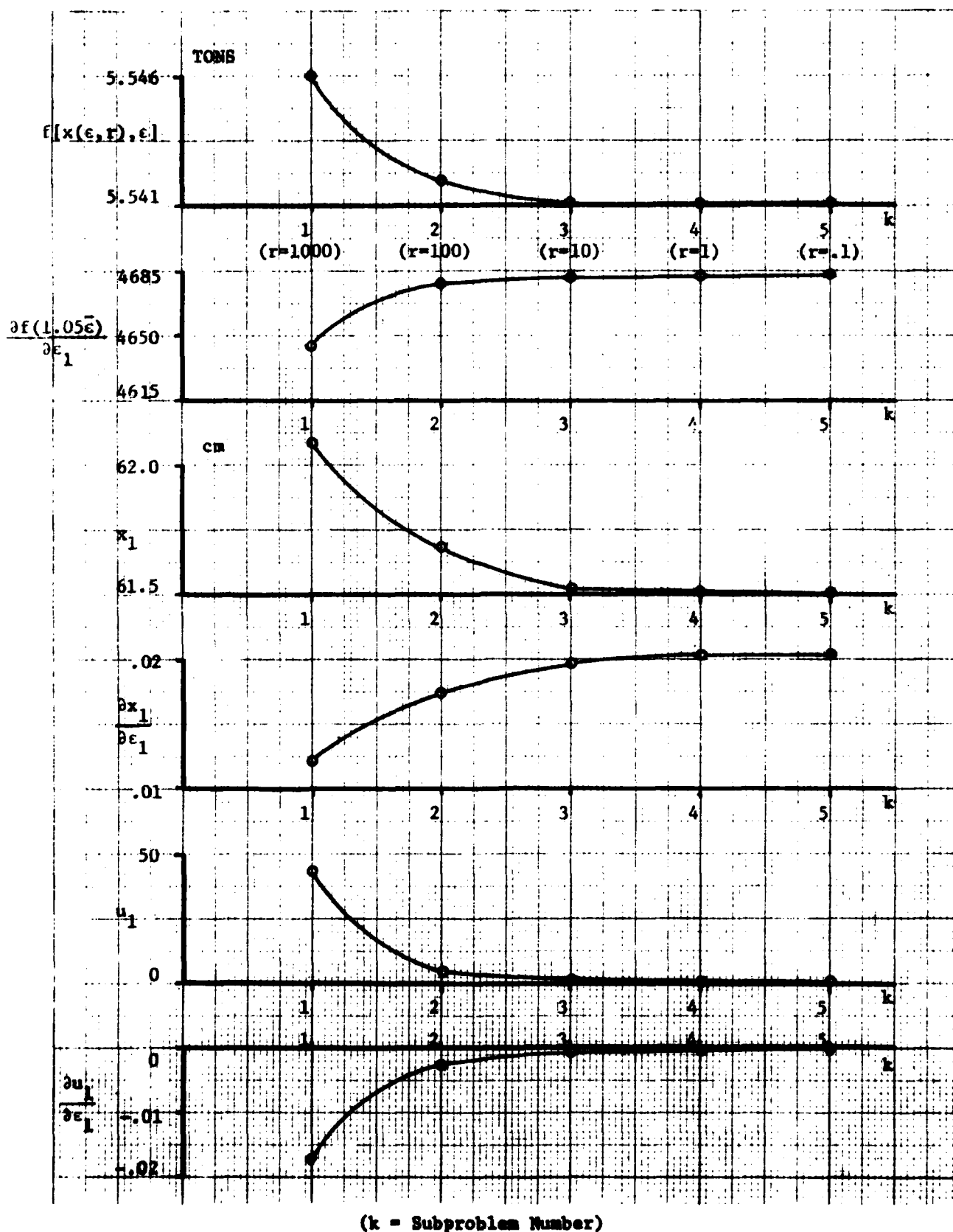


Figure 12. Sample of subproblem solution point and partial derivatives along the trajectory for Problem $P(1.05\bar{\epsilon})$.

6. FURTHER OBSERVATIONS AND CONCLUSIONS

As was noted, one of the principal uses of sensitivity analysis is the estimation of a solution of a perturbed problem, based on a solution and sensitivity information of the unperturbed problem. Obviously the "validity" of the extrapolated results depends on the magnitude of the perturbations introduced and, as true even in linear programming, extrapolations are generally valid only for a "limited" range of perturbations. A crucial difference is the fact that sharp bounds on this range can usually be determined in linear programming, while this is not usually true in nonlinear programming. In fact the state of the art is such that we are still developing techniques for calculating sharp error bounds for a specified measure of stability, an objective being to calculate such bounds in a computationally efficient manner. It is not satisfactory to require as much effort to calculate a sensitivity measure as it takes to solve the given problem from scratch. This is often not feasible in practice, nor does it appear to be generally essential in theory. A whole body of methodology has yet to be developed before useful stability calculations "in the large" are rigorously validated and efficiently implemented for general parametric nonlinear programming perturbation analysis.

Based on the results of the present analysis, the following conclusions are also offered, subject to the scenario described herein:

1. The weight of the corrugated bulkhead is extremely sensitive to changes in the minimum allowable panel thickness, which is in turn a function of the ship length. (This latter relationship is not present in the given model or pursued in this paper.) One millimeter relaxation of this parameter will save about 0.331 tons of steel per bulkhead (equivalent to 6.3 percent of optimal bulkhead weight).

A similar result holds for the corrosion allowance parameter k_2 . One millimeter relaxation of this parameter would save about 0.11 tons of steel per bulkhead (equivalent to about 2.1 percent of the optimal bulkhead weight).

2. Weight of the corrugated bulkhead is only partially sensitive to changes in the top and middle panel lengths and total panel depth. These sensitivities are completely overshadowed by those for minimum allowable thickness of the panels.
3. Surprisingly, the weight of the bulkhead is rather insensitive to changes in the parameter k_1 (a function of the allowable bending stress). A 50 kg/cm^2 relaxation on maximum allowable bending stress, which is a rather significant relaxation as far as design considerations are concerned, would save only about 0.0162 tons of steel per bulkhead (about 0.3 percent of optimal bulkhead weight).
4. Assuming the validity of the model, these results suggest that the most economical marginal reductions in bulkhead weight will be realized by first decreasing the minimum plate thickness requirement (t^{\min}) and next by relaxing the corrosion allowance (k_2).
5. For rather significant perturbations, the extrapolated solution of the perturbed problem, based on the solution and sensitivity information of the unperturbed problem is very accurate for the subject problem.

As a concluding remark, it is universally acknowledged that a sensitivity analysis is an indispensable part of a "solution," certainly in any practical context. Aside from providing invaluable insight into problem and model structure and estimates of the effects of changes in design or data parameters, a thorough analysis can provide warnings of instabilities, indications of improper problem or model formulations, and detailed guidelines for making cost-effective changes in the parameters.

The computer program SENSUMT is available to the general public on request to the authors.

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